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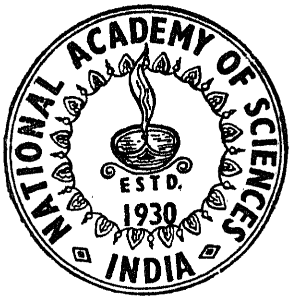

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Strong almost convergence of order (r) of the Fourier series

By

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1. Let $f(t)$ be a periodic function with period 2π and be integrable in the sense of Lebesgue over $(-\pi, \pi)$.

Let the Fourier series associated with the function $f(t)$ be

$$f(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{1·1}$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2S\}$$

and

$$F(t) = \int_0^t f(u)\, du.$$

2. In 1948, Lorentz [2] discussed a new process of summation which assigns a general limit Lim x_n to certain bounded sequences $x = \{x_n\}$. This method is analogous to the mean values which are used in the theory of almost periodic functions, furthermore, it is narrowly connected with limit of Banach [1]. The sequences which are summable by this method are called almost convergent sequences.

In this paper we shall first define a new method of summation which will be called "strong almost convergent" and also we shall find the sufficient conditions which sum the Fourier series by this method.

Definition : An infinite series Σa_n with the sequence of partial sum S_n is said to be strongly almost convergent of any positive order r to s., if

$$\sum_{\nu=0}^{n} | S_{\nu+p} - S |^r = o(n), \text{ as } n \to \infty \tag{2·1}$$

uniformly with respect to p.

The object of this paper is to prove the following.

Theorem : If p and q are conjugate indices, $p > 1$.

$$\int_0^t |\phi(u)|^p \, du = O\,(t.\log^{p/q} 1/t) \tag{2·2}$$

and

$$\int_0^t \phi(u)\, du = o\,(t/\log \frac{1}{t}), \text{ as } t \to o \tag{2·3}$$

Then

$$\sum_{\nu=0}^{n} | S_{\nu+k} - S |^r = o\,(n), \text{ as } n \to \infty,$$

Uniformly with respect to k.

3. **Proof of the Theorem :** With the usual standard simplifications, we write,

$$S_{n+k} = \frac{2}{\pi} \int_0^{\pi} f(u) \frac{\text{Sin}\,(n+k+\frac{1}{2})\,u}{2\,\text{Sin}\,\frac{1}{2}\,u}\, du$$

$$= \frac{2}{\pi} \left\{ \int_0^{\eta_1} + \int_{\eta_1}^{\eta_2} + \int_{\eta_2}^{\pi} \right\} \frac{\text{Sin}\,(n+k)\,u}{\tan \frac{1}{2}\,u} f(u)\, du +$$

$$+ \frac{2}{\pi} \int_0^{\pi} \text{Cos}\,(n+k)\,u\,f(u)\,du$$

$$= \frac{2}{\pi} \left\{ J_n^{(1)} + J_n^{(2)} + J_n^{(3)} + J_n^{(4)} \right\}, \tag{3·1}$$

say, where $\eta_1 = \dfrac{k_1 \log\,(n+k)}{n+k}$ and $\eta_2 = \dfrac{k_1 \log n}{n}$.

It is plain that $J_n^{(4)} \to o$, as $n \to \infty$, uniformly with respect to k and therefore

$$\left\{ \sum_{\nu=0}^{n} | J_\nu^{(4)} |^q \right\}^{1/q} = o\,(n^{1/q}), \text{ as } n \to \infty. \tag{3·2}$$

Now integrating by parts and using the condition (2·3) we get

$$J_n^{(1)} = \text{Sin}\,(n+k)\,\eta_1 \cot \tfrac{1}{2}\,\eta_1\, F(\eta_1) + o\,(1) -$$

$$-\int_0^{\eta_1} \left\{ (n+k) \cos\,(n+k)\,u \cot\frac{u}{2} - \tfrac{1}{2}\,\text{Sin}\,(n+k)\,u\,\text{cosec}^2\,\frac{u}{2} \right\} F(u)\,du$$

$$= o\,(1) + o\,(1).\int_0^{\eta_1} \left\{ \frac{n+k}{u} + \frac{(n+k)u}{u^2} \right\} . \left\{ \frac{u}{\log \frac{1}{u}} \right\} du$$

$$= o\,(1) + o\,(n+k)\int_{0}^{\eta_1} \frac{1}{\log \frac{1}{u}}\, du$$

$$= o(1), \text{ as } n \to \infty, \text{ unformly with respect to } k.$$

Thus,

$$\left\{ \sum_{\nu=0}^{n} |J_{\nu}^{(1)}|^{q} \right\}^{1/q} = o\,(n^{1/q}), \text{ as } n \to \infty, \tag{3·3}$$

uniformly with respect to k.

Further using the condition (2·3), we have

$$\begin{aligned} J_n^{(2)} &= \int_{\eta_1}^{\eta_2} f(u) \operatorname{Sin}\,(n+k)\,u \cot \tfrac{1}{2}\,u\, du \\ &\leqslant \int_{\eta_1}^{\eta_2} \frac{f(u)}{u}\, du \\ &= \left[\frac{F(u)}{u} \right]_{\eta_1}^{\eta_2} + \int_{\eta_1}^{\eta_2} \frac{F(u)}{u^2}\, du \\ &= o\,(1), \text{ as } n \to \infty, \end{aligned}$$

uniformly *w. r. t. k.*

Therefore

$$\left\{ \sum_{\nu=0}^{n} |J_{\nu}^{(2)}|^{q} \right\}^{1/q} = o\,(n^{1/q}), \text{ as } n \to \infty, \tag{3·4}$$

uniformly *w. r. t. k.*

Now we denote by $C_n(\tau)$ the n-th Fourier sine coefficient of the odd function $\chi(t)$ which is equal to $f(t)$ in (o, τ) and zero in (τ, π), thus we have,

$$J_n^{(3)} = \int_{\eta_2}^{\pi} \operatorname{Cot} \tfrac{1}{2}\, u \left\{ \frac{d}{dt} \int^{u} f(t) \operatorname{Sin}\,(n+k)\, t\, dt \right\} du$$

$$= -\tfrac{1}{2}\,\pi \cot \tfrac{1}{2}\,\eta_2\, C_{n+k}\,(\eta_2) + \frac{\pi}{4} \int_{\eta_2}^{\pi} \operatorname{cosec}^2 \frac{u}{2}\; C_{n+k}\,(u)\, du.$$

Using Minkowski's inequality, we write

$$\left\{ \sum_{\nu=0}^{n} |J_{\nu}^{(3)}|^{q} \right\}^{1/q} \leqslant \cot \tfrac{1}{2}\,\eta_2 \left\{ \sum_{\nu=0}^{n} |C_{\nu+k}\,(\eta_2)|^{q} \right\}^{1/q} +$$

$$+ \int_{\eta_2}^{\pi} \operatorname{cosec}^2 \tfrac{1}{2}\, u \left\{ \sum_{\nu=0}^{n} |C_{\nu+k}\;(u)|^{q} \right\}^{1/q} du.$$

Also by Hausdroff's inequality, we have

$$\left\{ \sum_{\nu=0}^{n} | C_{\nu+k}(u) |^{q} \right\}^{1/q} \leqslant \left\{ \int_{-\pi}^{\pi} | \chi(u) |^{p} du \right\}^{1/p}$$

$$= \left\{ \int_{-t}^{t} | f(u) |^{p} du \right\}^{1/p}$$

$$= o(1). \left\{ u^{1/p}. \log^{1/q} \frac{1}{u} \right\}.$$

Hence

$$\left\{ \sum_{\nu=0}^{n} | J_{\nu}^{(3)} |^{q} \right\}^{1/q} = O(1). \left\{ \frac{1}{\eta_2} \left(\eta_2^{1/p} \log^{1/q} \frac{1}{\eta_2} \right) \right\} +$$

$$+ O(1). \int_{\eta_2}^{\pi} \frac{1}{u^2} u^{1/p} \log^{1/q} \frac{1}{u} du.$$

$$= O(1). \left\{ \eta_2^{\frac{1}{p}-1} \log^{1/q} \frac{1}{\eta_2} \right\} +$$

$$+ O(1) \int_{\eta_2}^{\pi} u^{\frac{1}{p}-2} \log^{1/q} \frac{1}{u} du.$$

$$= O(1). \left\{ \eta_2^{-1/q}. \log^{1/q} \frac{1}{\eta_2} \right\}$$

(3·5) $$= o(n^{1/q}), \text{ as } n \to \infty,$$

uniformly *w. r. t. k.*

Finally collection of (3·1), (3·2), (3·3), (3·4) and (3·5) gives

$$\left\{ \sum_{\nu=0}^{n} | S_{\nu+k} |^{q} \right\}^{1/q} = o(n^{1/q}), \text{ as } n \to \infty,$$

uniformly *w. r. t. k.*

Hence

$$\sum_{\nu=0}^{n} | S_{\nu+k} |^{q} = o(n), \text{ as } n \to \infty,$$

uniformly *w. r. t. k.*

This completes the proof of the theorem.

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References

1. Banach, S. Theorie des operations linear, Warszwa, p. 33-34, 1932.
2. Lorentz, G. G. A contribution to the theory of divergent sequences, *Acta Mathematica,* **80**, 167-190, 1948.

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Some Theorems on Transform

By

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1. **Introduction :** In the present paper some theorems involving some functions and their transforms under any Fourier Kernel are obtained. The function $k(x)$ is said to be a Fourier Kernel if the following pair of reciprocal equations are simultaneously valid :

$$g(x) = \int_0^\infty k(xy)\, f(y)\, dy \tag{1}$$

$$f(x) = \int_0^\infty k(xy)\, g(y)\, dy \tag{2}$$

To ensure the validity of (1) and (2), the set of conditions we use here consists of convergence conditions on $f(x)$ and $k(x)$ together with a functional equation satisfied by the Mellin transform of $k(x)$. $K(s)$ is said to be the Mellin transform of $k(x)$ if

$$K(s) = \int_0^\infty k(x)\, x^{s-1}\, dx.$$

The functional equation satisfied by $K(s)$ is $K(s)\ K(1-s) = 1$. [5]

The result given in Titchmarsh's Fourier Integrals and Mitra's result [4] follow as particular cases of the theorems of this paper.

The theorems have been formulated under rather stringent conditions. Some of the conditions may be relaxed and analytic continuation method may be applied.

2. If $\theta(x)$ and $\phi(x)$ are two Fourier Kernels and if the integral

$\int_0^\infty \phi(t)\, \theta(xt)\, dt$ exists and is equal to $\phi_\theta(x)$, say, then $\phi_\theta(x)$ is also a Fourier Kernel [5]

We shall require the following two throrems [5] :

I. Let $y^{c-1} f(y)$ belong to $L(o, \infty)$ and let $f(y)$ be continuous and of bounded variation in the neighbourhood of the point $y = x$. Let

$$F(s) = \int_0^\infty f(x)\, x^{s-1}\, dx,\ s = c + it. \quad \text{Then}$$

$$J(s) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} F(s)\, x^{-s}\, ds.$$

II. Let $F(c+iu)$ belong to $L(-\infty, \infty)$ and let it be continuous and of bounded variation in the neighbourhood of the point $u = t$. Let

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s)\, x^{-s}\, dx. \quad \text{Then}$$

$$F(s) = \lim_{T \to \infty} \int_{\frac{1}{T}}^{T} f(x)\, x^{s-1}\, dx \quad \text{for } s = c + it.$$

In our theorems we shall assume that the conditions mentioned in these two theorems hold.

3. We shall adopt the following notations :

$$\phi_\theta(x) = \int_0^\infty \phi(t)\, \theta(xt)\, dt \quad \text{and}$$

$$(\phi o \theta)(x) = \int_0^\infty \theta(xt)\, \phi\left(\frac{1}{t}\right) \frac{dt}{t},$$

where $\theta(x)$ and $\phi(x)$ are Fourier Kernels continuous in each arbitrary finite interval (O, X) and the integrals converge uniformly in (O, X), so that $\phi_\theta(x)$ exist and are continuous.

If $L(s)$ and $K(s)$ are the Mellin transforms of $\theta(x)$ and $\phi(x)$ respectively, then it is easy to see that $L(s)\ K(s)$ is the Mellin transform of $(\theta o \phi)(x)$ and $(\theta o \phi)(x) = (\phi o \theta)(x)$ and $(\theta o \phi)(x)$ is also a Fourier Kernel.

We may remark here that if θ, ϕ, β are three Fourier Kernels then

$$(\theta o\, (\phi o \beta))(x) = ((\theta o \phi)\, o \beta)(x) = (\theta o \phi o \beta)(x).$$

We may extend this result to n Fourier kernels. Let $\theta_1, \theta_2, \ldots, \theta_n$ be n Fourier kernels. Then $(\theta_1\, o\, \theta_2\, o \ldots o\, \theta_n)(x)$ is also a Fourier kernel. The θ's may be permuted among themselves, and all θ's are continuous in each arbitrary finite interval (o, X) and the integrals converge uniformly. And

$$\begin{aligned}
&(\theta_1 o \theta_2 o \ldots o \theta_n)(x) \\
&= \int_0^\infty \cdots \int_0^\infty \theta_1(t_1)\, \theta_2(t_2) \ldots \theta_{n-1}(t_{n-1})\, \theta_n\left(\frac{x}{t_1\, t_2 \ldots t_{n-1}}\right) \times \\
&\qquad \times \frac{dt_1\, dt_2 \ldots dt_{n-1}}{t_1\, t_2 \ldots t_{n-1}}.
\end{aligned}$$

If we take

$$\theta_1(x) = \sqrt{x}\, J_{\nu_1}(x),\ \theta_2(x) = \sqrt{x}\, J_{\nu_2}(x), \ldots,$$

$$\theta_n(x) = \sqrt{x}\, J_{\nu_n}(x), \text{ then}$$

$$(\theta_1 o \theta_2 o \ldots o \theta_n)(x)$$

$$= \sqrt{x} \int_0 J_{\nu_1}(t_1)\, J_{\nu_2}(t_2) \ldots J_{\nu_{n-1}}\ {}_{n-1})\, J_{\nu_n}\left(\frac{x}{t_1\, t_2 \ldots t_{n-1}}\right) \times$$

$$\times \frac{dt_1 \ldots dt_{n-1}}{t_1 \ldots t_{n-1}}$$

$= \varpi_{\nu_1, \ldots, \nu_n}(x)$, a Forier Kernel introduced by Bhatnagar [1]

In the theorems proved in the present paper, all the θ's need not involve Bessel Functions of the first kind or sine or cosine functions. Recently Fox [3] has introduced a Kernel and our theorems remain true if we replace some or all the θ's by Fox's Kernel provided the convergence of the integrals is secured and other conditions mentioned before hold.

We shall also adopt the notation :

$$f_\theta(x) = \int_0^\infty f(t)\, \theta(xt)\, dt.$$

Provided $f(t)$ belongs to $L(o, \infty)$ and $\theta(x)$ is bounded and integrable in an arbitrary interval, $f_\theta(x)$ always exists. We note that $f_\theta(x)$ is the Fourier Transform of $f(t)$ with regard to the Kernel $\theta(t)$.

4. We next prove a few theorems.

THEOREM 1 : Let $f(x)$ and $g(x)$ be two functions continuous and belonging to $L(o, \infty)$. Then

$$\int_0^\infty f_\theta(x)\, g_\theta(x)\, dx = \int_0^\infty f(x)\, g(x)\, dx.$$

This is known theorem.

THEOREM 2 : Let $f(x)$ and $g(x)$ be two continuous functions and $\theta(x)$, $\phi(x)$ two Fourier Kernels such that $\theta(x)$, $\phi(x)$ are continuous in each arbitrary finite interval (o, X) and the integral involving the Kernels *i.e.*, $(\theta o \phi)(x)$ converge uniformly. Then

$$\int_0^\infty f_{\theta o \phi}(t)\, g_{\theta o \phi}(t)\, dt = \int_0^\infty f_\theta(t)\, g_\theta(t)\, dt$$

$$= \int_0^\infty f_\phi(t)\, g_\phi(t)\, dt$$

$$= \int_0^\infty f(t)\, g(t)\, dt.$$

Before proving the theorem we prove the following lemma.

LEMMA : $f_{\theta o \phi}(x) = \left(\frac{1}{x} f_\theta\left(\frac{1}{x}\right)\right)_\phi = \left(\frac{1}{x} f_\phi\left(\frac{1}{x}\right)\right)_\theta.$

PROOF : We know that

$$f_{\theta o \phi}(x) = \int_0^\infty f(t)\,dt \int_0^\infty \theta\left(\frac{1}{u}\right) \phi\,(xtu)\,\frac{du}{u}$$

$$= \int_0^\infty \theta\left(\frac{1}{u}\right)\frac{du}{u}\int_0^\infty f(t)\,\phi\,(xtu)\,dt$$

$$= \int_0^\infty \theta\left(\frac{x}{u}\right)\frac{du}{u}\int_0^\infty f(t)\,\phi(tu)\,dt$$

$$= \int_0^\infty \theta\left(\frac{x}{u}\right) f_{\phi}(u)\,\frac{du}{u}$$

$$= \int_0^\infty \theta(xu)\,f_{\phi}\left(\frac{1}{u}\right)\frac{du}{u}$$

$$= \left(\frac{1}{x} f_{\phi}\left(\frac{1}{x}\right)\right)_{\theta} .$$

Similarly we can show that

$$f_{\theta o \phi}(x) = \left(\frac{1}{x} f_{\theta}\left(\frac{1}{x}\right)\right)_{\phi} .$$

Thus the lemma is proved.

Proof of the theorem :

$$\int_0^\infty f_{\theta o \phi}(x)\, g_{\theta o \phi}(x)\,dx$$

$$= \int_0^\infty \left(\frac{1}{x} f_{\theta}\left(\frac{1}{x}\right)\right)_{\phi} \left(\frac{1}{x} g_{\theta}\left(\frac{1}{x}\right)\right)_{\phi} dx$$

$$= \int_0^\infty \frac{1}{x} f_{\theta}\left(\frac{1}{x}\right) \frac{1}{x} g_{\theta}\left(\frac{1}{x}\right) dx$$

$$= \int_0^\infty f_{\theta}(x)\, g_{\theta}(x)\,dx$$

$$= \int_0^\infty f_{\phi}(x)\, g_{\phi}(x)\,dx$$

$$= \int_0^\infty f(x)\, g(x)\,dx.$$

COROLLARY 1 : Let $f(x)$ and $g(x)$ be two continuous functions belonging to $L(o, \infty)$ and $\theta_1(x)$, $\theta_2(x)$, . . . , $\theta_n(x)$ be n Fourier Kernels such that $\theta_r(x)$, $r = 1, 2, \ldots, n$, is continuous in each arbitrary finite interval (o, X) and the integrals involving the Kernels *i.e.*, $(\theta_1 o \theta_2 o \ldots o\, \theta_r)\,(x)$, converge uniformly. Then

$$\int_0^\infty f_{\theta_1 o\theta_2 o \dots o\theta_n}(x)\, g_{\theta_1 o\theta_2 o \dots o\theta_n}(x)\, dx$$

$$= \int_0^\infty f_{\theta_r}(x)\, g_{\theta_r}(x)\, dx, \; (r = 1, 2, \dots, n)$$

$$= \int_0^\infty f(x)\, g(x)\, dx.$$

PROOF : Since the θ's can be permuted among themselves

$$(\theta_1 o\theta_2 o \dots o\theta_n)(x) = (\theta_r o(\theta_1 o\theta_2 o \dots o\, \theta_{r-1}\, o\, \theta_{r+1} o \dots o\theta_n))(x).$$

Now $(\theta_1 o\theta_2 o \dots o\theta_{r-1} o\theta_{r+1} o \dots o\theta_n)(x)$ is a Fourier Kernel continuous in each arbitrary finite interval (o, X) (the integrals converging uniformly). Let us denote this Kernel by $\psi(x)$.

Therefore

$$(\theta_1 o\theta_2 o \dots o\theta_n)(x) = (\theta_r o\psi)(x).$$

Now

$$\int_0^\infty f_{\theta_1 o\theta_2 o \dots o\,\theta_n}(x)\, g_{\theta_1 o \dots o\theta_n}(x)\, dx$$

$$= \int_0^\infty f_{\theta_r o\psi}(x)\, g_{\theta_r o\psi}(x)\, dx$$

$$= \int_0^\infty f_{\theta_r}(x)\, g_{\theta_r}(x)\, dx$$

$$= \int_0^\infty f(x)\, g(x)\, dx.$$

COROLLARY 2 : Under the conditions of corollary 1,

$$\int_0^\infty f_{\theta_1 o \dots o\theta_n}(x)\, g_{\theta_1 o \dots o\theta_n}(x)\, dx$$

$$= \int_0^\infty f_{\theta_1 o\, {}_2 o \dots o\theta_r}(x)\, g_{\theta_1 o\theta_2 o \dots o\theta_r}(x)\, dx$$

$$= \int_0^\infty f_{\theta_{r+1} o\, \theta_{r+2} o \dots o\theta_n}(x)\, g_{\theta_{r+1} o\theta_{r+2} o \dots o\theta_n}(x)\, dx$$

$$= \int_0^\infty f(x)\, g(x)\, dx, \text{ where } r = 1, 2, \dots, n-1.$$

COROLLARY 3 :

$$\int_0^\infty f_\theta(x)\, g_{\theta o\phi}(x)\, dx = \int_0^\infty f\left(\frac{1}{t}\right)\frac{1}{t}\, g_\phi(t)\, dt,$$

provided $f(t)$ and $g(t)$ are continuous and belong to $L(o, \infty)$ and $\theta(x)$, $\phi(x)$ are continuous in (o, X) for all finite values of X and the integral for $(\theta o\phi)(x)$ cooverages uniformly, and the integral exist.

PROOF : This can be proved by comparing the Mellin Transforms of both sides. We also get the following generalised result :

$$\int_0^\infty f_{\theta_1 o \theta_2 o \dots o \theta_r}(t)\; g_{\theta_1 o \theta_2 o \dots o \theta_r o \theta_{r+1} o \dots o \theta_n}(t)\, dt$$

$$= \int_0^\infty f\left(\frac{1}{t}\right) g_{\theta_{r+1} o \dots o \theta_n}(t) \frac{dt}{t},$$

under the conditions of the theorem.

In corollary 1, putting $(\theta_1 o \theta_2 o \dots o \theta_n)(x) = \varpi_{\nu_1, \dots, \nu_n}(x)$ we get theorem 2 of Mitra [4]. In corollary 2, putting $r = 1, 2, \dots, n-1$, in succession we get the corollary 1 of Mitra.

We note here that in all these theorems a change in the order of integration is permissible by De-la-Vallee Poussin's Theorem [2].

5. Let

(*i*) $\theta(x)$ and $\phi(x)$ be continuous functions of x in (o, X) for all finite values of X and let the integral involving θ and ϕ, *viz.* $\phi_\theta(x)$ converge uniformly.

(*ii*) Let $f(x)$, $g(x)$ and $h(x)$ be continuous and belong to $L(o, \infty)$.

(*iii*) $\phi_\theta(x)$ be bounded and integrable in (o, ∞).

(*iv*) $f_\theta(x)$, $g_\phi(x)$ and ${}^h\phi_\theta(x)$ be absolutely integrable in (o, ∞), where

$${}^h\phi_\theta(x) = \int_0^\infty h(t)\, \phi_\theta(xt)\, dt.$$

Then

THEOREM 3 :

$$\int_0^\infty \int_0^\infty {}^h\phi_\theta(x_1)\, f_\theta(x_1 x_2)\, g_\phi(x_2)\, dx_1\, dx_2$$

$$= \int_0^\infty \int_0^\infty h(t_1)\, f(t_1 t_2)\, g(t_2)\, dt_1\, dt_2$$

PROOF : Now

$$\int_0^\infty \int_0^\infty {}^h\phi_\theta(x_1)\, f_\theta(x_1 x_2)\, g_\phi(x_2)\, dx_1\, dx_2$$

$$= \int_0 {}^h\phi_\theta(x_1)\, dx_1 \int_0^\infty f_\theta(x_1 x_2)\, g_\phi(x_2)\, dx_2$$

$$= \int_0^\infty {}^h\phi_\theta(x_1)\, dx_1 \int_0^\infty \left\{ \int_0^\infty f(t_1)\, \theta(x_1 x_2 t_1)\, dt_1 \int_0^\infty g(t_2)\, \phi(x_2 t_2)\, dt_2 \right\} dx_2$$

$$= \int {}^h\phi_\theta(x_1)\, dx_1 \int_0^\infty f(t_1)\, dt_1 \int_0^\infty g(t_2)\, dt_2 \int_0^\infty \theta(x_1 x_2 t_1)\; \phi(x_2 t_2)\, dx_2$$

$$= \int_0^\infty h_{\phi_\theta}(x_1)\, dx_1 \int_0^\infty f(t_1)\, dt_1 \int_0^\infty g(t_2)\, \frac{dt_2}{t_2} \int_0^\infty \theta\left(\frac{x_1 x_2 t_1}{t_2}\right) \phi(x_2)\, dx_2$$

$$= \int_0^\infty h_{\phi_\theta}(x_1)\, dx_1 \int_0^\infty f(t_1)\, dt_1 \int_0^\infty g(t_2)\, \phi\theta\left(\frac{x_1 t_1}{t_2}\right) \frac{dt_2}{t_2}$$

$$= \int_0^\infty f(t_1)\, dt_1 \int_0^\infty g(t_2)\, \frac{dt_2}{t_2} \int_0^\infty h_{\phi_\theta}(x_1)\, \phi\theta\left(\frac{x_1 t_1}{t_2}\right) dx_1$$

$$= \int_0^\infty f(t_1)\, dt_1 \int_0^\infty g(t_2)\, h\left(\frac{t_1}{t_2}\right) \frac{dt_2}{t_2}$$

$$= \int_0^\infty g(t_2)\, dt_2 \int_0^\infty f(t_1\, t_2)\, h(t_1)\, dt_1$$

$$= \int_0^\infty \int_0^\infty h(t_1)\, f(t_1\, t_2)\, g(t_2)\, dt_1\, dt_2.$$

The changes in the order of integrations are permissible by the conditions of the theorem.

THEOREM 4 : Under the same conditions

$$\int_0^\infty \int_0^\infty h(t_1)\, f_\theta(t_1\, t_2)\, g_\phi(t_2)\, dt_1\, dt_2$$

$$= \int_0^\infty \int_0^\infty h_{\phi_\theta}(t_2)\, f(t_1\, t_2)\, g(t_2)\, dt_1\, dt_2.$$

PROOF : This can be proved in a similar manner.

THEOREM 5 : Let

(*i*) $\theta(x)$ and $\phi(x)$ be continuous in (o, X) for all finite values of X.

(*ii*) Let $f(x)$, $g(x)$, $\frac{1}{x} g(x)$ and $h(x)$ be continuous and belong to $L(o, \infty)$.

(*iii*) $(\phi\ o\ \theta)\ (x)$ be bounded and integrable in (o, ∞).

(*iv*) $f_\theta(x)$, $\frac{1}{x}\, G\left(\frac{1}{x}\right)$ and $h_{\phi o \theta}(x)$ be absolutely integrable in (o, ∞), where

$$G(x) = \int_0^\infty g(t)\, \phi\left(\frac{x}{t}\right) \frac{dt}{t}$$

Then

$$\int_0^\infty \int_0^\infty h_{\phi o \theta}(x_1)\, f_\theta(x_1\, x_2)\, G\left(\frac{1}{x_2}\right) \frac{dx_1\, dx_2}{x_2}$$

$$= \int_0^\infty \int_0^\infty h(t_1)\, f(t_1\, t_2)\, g(t_2)\, dt_1\, dt_2.$$

PROOF :

$$\int_0^\infty \int_0^\infty h_{\phi o\theta}(x_1)\ f_\theta(x_1x_2)\ G\left(\frac{1}{x_2}\right)\frac{dx_1\,dx_2}{x_2}$$

$$=\int_0^\infty h_{\phi o\theta}(x_1)\ dx_1 \int_0^\infty \left\{ \int_0^\infty f(t_1)\ \theta(x_1x_2t_1)\ dt_1 \int_0^\infty g(t_2)\ \phi\left(\frac{1}{x_2\,t_2}\right)\frac{dt_2}{t_2}\right\}\frac{dx_2}{x_2}$$

$$=\int_0^\infty h_{\phi o\theta}(x_1)\ dx_1 \int_0^\infty f(t_1)\ dt_1 \int_0^\infty g(t_2)\ \frac{dt_2}{t_2}\int_0^\infty \theta(x_1x_2t_1)\ \phi\left(\frac{1}{x_2t_2}\right)\frac{dx_2}{x_2}$$

$$=\int_0^\infty h_{\phi o\theta}(x_1)\ dx^1\int_0^\infty f(t_1)\ dt_1 \int_0^\infty g(t_2)\ \frac{dt_2}{t_2}\int_0^\infty \theta\ \left(\frac{x_1x_2t_1}{t_2}\right)\ \phi\ \left(\frac{1}{x_2}\right)\frac{dx_2}{x_2}$$

$$=\int_0^\infty h_{\phi o\theta}(x_1)\ dx_1 \int_0^\infty f(t_1)\ dt_1 \int_0^\infty g(t_2)\ \frac{1}{t_2}\ (\phi o\theta)\left(\frac{x_1t_1}{t_2}\right)\ dt_2$$

$$=\int_0^\infty f(t_1)\ dt_1 \int_0^\infty g(t_2)\ \frac{1}{t_2}dt_2 \int_0^\infty h_{\phi o\theta}(x_1)\ (\phi o\vartheta)\left(\frac{x_1t_1}{t_2}\right)\ dx_1$$

$$=\int_0^\infty f(t_1)\ dt_1 \int_0^\infty g(t_2)\ \frac{1}{t_2}\ h\left(\frac{t_1}{t_2}\right)\ dt_2$$

$$=\int_0^\infty h(t_1)\,f(t_1t_2)\,g(t_2)\ dt_1\ dt_2.$$

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References

1. Bhatnagar, K. P. (*i*) *Acad. Royal Belgique Bull. cl. Sci.*, **39**(5), 42-69, 1953. (*ii*) *Thesis for the Ph.D. degree, Lucknow University*, 1953.
2. Carslaw, H. S. Theory of Fourier's Series and Integrals. *3rd ed. Dover Publications*, 1930.
3. Fox, C. *Trans. Amer. Math. Soc.*, **98**, 1961.
4. Mitra, S. C. *Canad. Math. Bull.*, **10**(5), 1967.
5. Titchmarsh, E. C. Theory of Fourier Integrals, 2*nd ed. Oxford*, 1948.

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Convolution in a General Transform

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Abstract

In this paper Convolution of two functions have been expressed as an integral of transforms of these functions with respect to a pair of inverse transforms. A result of Mellin transform in terms of fractional integral is illustrated by examples.

Introduction

1. Convolution of two functions $f(t)$ and $g(t)$ is defined by Erde'lyi [4, page 8]

$$f * g = \int_0^t f(u)\, g(t-u)\, du \qquad t \geqslant o. \tag{1·1}$$

for the interval (o, t)

Ruel. V. Churchill [3, page 274] has defined it over the interval $(-\pi, \pi)$. Convolution or Faltung of the functions f and g over the interval $(-\infty, \infty)$ is expressed as Fourier Transforms of functions f and g [5, page 24] as

$$f * g = \int_{-\infty}^{\infty} f(x)\, g(u-x)\, dx = \int_{-\infty}^{\infty} F(t)\, G(t)\, e^{-iut}\, dt \tag{1·2}$$

Similar results for intervals (o, x) and (x, ∞) are known [5, page 31] for Laplace transforms.

Convolution of two functions over different intervals have been expressed in terms of integrals involving transforms of the functions f and g.

The object of this paper is to express the Convolution of two functions f and g as an integral involving transforms of these functions. This will be done by establishing a theorem. Some particular cases will give known results while others are believed to be new. The theorem is illustrated by two examples. The approach is formal.

2. *Theorem* If

(*i*) $k_1(x)$ and $k_2(x)$ are kernels of a transform and its inverse, having C_1 and C_2 as the limits of integration so that

$$F(y) \equiv K_1\{f(x); y\} = \int_{C_1} f(x)\, k_1(xy)\, dx. \tag{2·1}$$

$$f(x) \equiv K_2\{F(y); x\} = \int_{C_2} F(y)\, k_2(xy)\, dy. \tag{2·2}$$

(*ii*) $f(t)\, g(x-t)$ is integrable over the interval C_1

(*iii*) the integral

$$\int_{C_2} K_1 \{ g(\xi) \,; y \} \, k_2 \, (yx) \, dy$$

is absolutely convergent

then

the convolution of two functions $f(t)$ and $g(t)$ over the interval C_1 is given by

$$(2{\cdot}3) \qquad \int_{C_1} f(t) \, g(x-t) \, dt = \int_{C_2} G(y) \, r(x,y) \, dy$$

where

$$(2{\cdot}4) \qquad G(y) = K_1 \{ g(\xi) \,; y \} \text{ and } r(x,y) = \int_{C_1} f(t) \, k_2 \, \{ (x-t) \, y \} \, dt$$

Proof: Because of (i) we can write

$$(2{\cdot}5) \qquad g(x-t) = \int_{C_2} K_1 \{ g(\xi) \,; y \} \, k_2 \{ (x-t) \, y \} \, dy$$

Using (2·5) in L. H. S. of (2·3) we get

$$\int_{C_1} f(t) \; g(x-t) \; dt = \int_{C_2} f(t) \left[\int_{C_2} K_1 \, \{ g(\xi) \,; y \} \, k_2 \, \{ (x-t) \, y \} \, dy \right] dt.$$

Changing the order of integration which is permissible under the conditions of the theorem we have

$$\int_{C_1} f(t) \, g(x-t) \, dt = \int_{C_2} K_1 \, \{ g(\xi) \,; y \} \left[\int_{C_1} f(t) \, k_2 \, \{ (x-t) \, y \} \, dt \right] dy.$$

Using (2·4) we arrive at (2·3).

3. *Particular Cases*: In cases where $k_2 \, \{ (x-t) \, y \}$ can be broken up into factors of the type $k_2 \, (xy) \, k_1 \, (yt)$, the function $r(x,y)$ is further simplified. This can be seen in the cases of Fourier transform, Laplace transform and Mellin Inverse transform. Other cases of Mellin transform, Fourier Cosine, Fourier Sine are believed to be new.

(1) Fourier Transform. Let $k_1 \, (x) = e^{ix}$, $k_2 \, (x) = e^{-ix}$, $C_1 = C_2 = (-\infty, \infty)$ then $r(x,y)$ of (2·4) becomes $e^{-ixy} \, F(y)$ and (2·3) reduces to the known result [5, page 24].

$$\int_{-\infty}^{\infty} f(t) \; g(x-t) \; dt = \int_{-\infty}^{\infty} G(y) \, F(y) \, e^{-ixy} \, dy.$$

Interchanging $k_1(x)$ and $k_2 \, (x)$ we have

$$\int_{-\infty}^{\infty} G(x-t) \; F(t) \, dt = \int_{-\infty}^{\infty} f(y) \, g(y) \, e^{ixy} \, dy.$$

(ii) Laplace Transform. Let $k_1(x) = e^{-px}$, $k_2 \, (p) = \dfrac{e^{px}}{2\pi i}$ C_1 is (o, ∞) and C_2 is $(c - i \infty$ to $c + i \infty)$ then $r(x,y) = e^{xy} \, F(y)$ and (2·3) gives known result [5, page 31]. Interchanging $k_1 \, (x)$, C_1 and $k_2 \, (x)$, C_2 in (2·3) we have

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(x-t)\, F(t)\, dt = \int_{o}^{\infty} g(y) f(y)\, e^{-xy}\, dy.$$

(*iii*) Mellin Inverse. Let $k_1(sx) = x^{-s}/2\pi i$, $k_2(sx) = x^{s-1}$ C_1 is $(c - i\infty$ to $c + i\infty)$, C_2 is $(o$ to $\infty)$, then $r(x, y) = y^{x-1} f(y)$ and (2·3) assumes a known form [6, page 52].

(*iv*) Mellin Transform. Let $k_1(xs) = x^{s-1}$, $k_2(sx) = x^{-s}/2\pi i$

Now

(3·1) $$g(x-t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(y)\, (x-t)^{-y}\, dy$$

and this exists for $t < x$

Modyfying accordingly

$$r(x, y) = \Gamma(1-y)\, I_x^{(1-y)} f(t). \qquad o < y < 1$$

where $I_x^{(1-y)} f$ denotes the operator of Reimann-Liouville fractional integral operator of order $(1-y)$ on $f(t)$.

Ultimately (2·3) becomes

(3·2) $$\int_0^x f(t)\, g(x-t)\, dt = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} G(y)\, \Gamma(1-y)\, I_x^{(1-y)} f(t)\, dy.$$

$$o < y < 1$$

Above result can be easily verified by taking

$$f(t) = t^{\nu-1} \quad \text{and} \quad g(t) = e^{-t},\ Re\ \nu > o.$$

(*v*) Fourier Cosine Transform. Let $k_1(x) = k_2(x) = \sqrt{\frac{2}{\pi}} \cos x$, $F_c(y)$ and $G_c(y)$ denote the Fourier Cosine transforms of $f(t)$ and $g(t)$ and let

$$F_{s,x}(y) = \int_0^x f(t) \sin(ty)\, dt$$

and

$$F_{c,x}(y) = \int_0^x f(t) \cos(ty)\, dt$$

then

$$r(x, y) = \sqrt{\frac{2}{\pi}} \cos(xy)\, F_{c,x}(y) + \sqrt{\frac{2}{\pi}} \sin(xy)\, F_{s,x}(y)$$

Thus theorem becomes

(3·3) $$\int_0^x f(t)\, g(x-t)\, dt = \sqrt{\frac{2}{\pi}}\int_0^\infty G_c(y)\, F_{c,x}(y) \cos(xy)\, dy$$
$$+ \sqrt{\frac{2}{\pi}}\int_0^\infty G_c(y)\, F_{s,x}(y) \sin(xy)\, dy.$$

(*vi*) Similar result for Fourier Sine transform will be

$$\int_0^x f(t)\ g(x-t)\ dt = \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(y)\ F_{c,x}(y)\ \mathrm{Sin}\ (xy)\ dy \tag{3·4}$$

$$- \sqrt{\frac{2}{\pi}} \int_0^\infty G_s(y)\ F_{s,x}\ (y)\ \mathrm{Cos}(xy)\ dy.$$

Above results can be verified by taking functions

$$f\ (t) = e^{-\alpha t},\ g(t) = e^{-\beta t},\ \alpha > \beta > o.$$

4. *Example* 1 : Let $f\ (t) = \mathrm{Cos}\ (at),\ g(t) = e^{-\beta t}$

Using the result [2, page 189 Vol. II].

$$I_x^{(1-y)}\ \mathrm{Cos}\ (at) = \frac{x^{1-y}}{2\ \Gamma(2-y)} \Big[{}_1F_1\ (1\ ;\ 2-y\ ;\ iax) + {}_1F_1\ (1\ ;\ 2-y\ ;\ -iax) \Big] \quad 1-y > o. \tag{4·1}$$

Using [1, page 312, Vol. I]

Mellin transform of $e^{-\beta t}$ is

$$G\ (y) = \frac{\Gamma(y)}{\beta^y}\ y > o. \tag{4·2}$$

Substituting these values in (3·2) we have

$$\int_0^x \mathrm{Cos}\ (at)\ e^{-\beta(x-t)}\ dt = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(y)\ \Gamma(1-y)\ x^{1-y}}{\beta^y\ 2\ \Gamma(2-y)}$$

$$\times \Big[{}_1F_1\ (1\ ;\ 2-y\ ;\ iax) + {}_1F_1\ (1\ ;\ 2-y;\ -iax) \Big] dy$$

Evaluating the L. H. S. and simplifying the R. H. S. we get

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma\ (y)}{(1-y)} \Big[{}_1F_1\ (1\ ;\ 2-y\ ;\ iax) + {}_1F_1\ (1\ ;\ 2-y\ ;\ -iax) \Big]\ x^{-y}\ \beta^{-y}\ dy$$

$$= \frac{2}{x(a^2+\beta^2)} \Big\{ \beta\ \mathrm{Cos}\ (ax) + a\ \mathrm{Sin}\ (ax) - \beta e^{-\beta x} \Big\} \quad o < y < 1,\ \ |\beta| < |a|$$

Replacing β by z, x by λ, we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(y)}{(1-y)} \Big[{}_1F_1\ (1\ ;\ 2-y\ ;\ ia\lambda) + {}_1F_1\ (1\ ;\ 2-y\ ;-ia\lambda) \Big] \lambda^{-y}\ z^{-y}\ dy$$

$$= \frac{2}{\lambda\ (a^2+z^2)} \Big\{ z\ \mathrm{Cos}\ (a\lambda) + a\ \mathrm{Sin}\ (a\lambda) - z\ e^{-\lambda z} \Big\} \quad o < Re\ y < 1,\ \ |z| < |a|$$

This result can be verified by evaluating the integral with the help of the residue theorem, the contour being a line parallel to imaginary axis, cutting the real axis at c such that $o < c < 1$, encircling the poles *viz.* $(o, -1, -2 \ldots\ldots)$

Example 2 : Let $f(t) = e^{\alpha t}$, $g(t) = \text{Sin}\ \beta t$, then using [2, page 187].

$$(4{\cdot}3) \qquad I_x^{(1-y)} f = \frac{x^{1-y}}{\Gamma(2-y)}\ {}_1F_1 \left\{ 1\ ; 2-y\ ; \alpha\, x \right\} \qquad 1-y > o$$

also using [1, page 317] Mellin transform of $g(t)$, becomes

$$(4{\cdot}4) \qquad G(y) = \frac{\Gamma(y)}{\beta^y}\ \text{Sin} \left(\frac{\pi}{2} y \right) \quad -1 < Re\, y < 1$$

Substituting these values in (3·2) and simplifying we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(y)}{(1-y)}\ \text{Sin} \left(\frac{\pi}{2} y \right) {}_1F_1 \left\{ 1\ ;\ 2-y\ ; \alpha\, x \right\} x^{-y}\, \beta^{-y}\, dy$$

$$= \frac{\beta\ (e^{\alpha x} - \text{Cos}\ \beta x) - \alpha\ \text{Sin}\ \beta x}{x\ (\alpha^2 + \beta^2)} \qquad o < Re\, y < 1, \ \ |\beta| < |\alpha|$$

Replacing β by z, x by λ, we have

$$\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(y)}{(1-y)}\ \text{Sin} \left(\frac{\pi}{2} y \right)\ {}_1F_1 \left\{ 1\ ; 2y\ ; \alpha\, \lambda \right\} \lambda^{-y}\, z^{-y}\, dy$$

$$= \frac{z\ (e^{\alpha\lambda} - \text{Cos}\ \lambda z) - \alpha\ \text{Sin}\ \lambda z}{\lambda\ (\alpha^2 + z^2)} \qquad o < Re\, y < 1, \ \ |z| < |\alpha|$$

This example can also be verified as in Ex. 1.

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References

1. Bateman Manuscript Project. Tables of Integral Transform. *McGraw Hill*, **Vol. I**, 1954.
2. Bateman Manuscript Project. Tables of Integral transform. *McGraw Hill*, **Vol. II**, 1954.
3. Churchill, Ruel V. Modern Operational Mathematics in Engineering. *McGraw Hill*, 1944.
4. Erde'lyi, Arthur. Operational Calculus and Generalised Functions, 1961.
5. Sneddon, I. N. Fourier Transforms. *McGraw Hill*, 1951.
6. Titchmarsh, E. C. Fourier Integral, 1937.

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Superposition of Perfect Fluid and Electromagnetic Fields in Sperically Symmetric Space–Time in General Relativity

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Abstract

In obtaining solutions of the field equations of general relativity the energy–momentum tensor employed is either that of material distribution or of electromagnetic distribution. In this paper we examine whether the superposition of the two fields is consistent within the frame-work of general relativity. It has been shown that such a superposition is possible in case when the material field consists of perfect fluid distribution and the space-time is spherically symmetric. In the special case of the distribution of discrete particles the density and the electromagnetic energy have been expressed in terms of the metric potentials.

Introduction

The form of the material energy tensor in general relativity in case of perfect fluid distribution is given by

$$T_j^i = (\varepsilon + p)\, u^i u_j + \delta_j^i p, \quad u^i u_i = -1, \tag{1·1}$$

where ε, p, u^i are respectively the matter density, pressure and the unit flow vector. On one hand when $p = 0$ it leads to a distribution of discrete particles and on the other, when $T = T_i^i = 0$ it gives the photon fluid distribution corresponding to the maximum pressure-density ratio, *viz.* $p/\varepsilon = \frac{1}{3}$. Other cases of perfect fluid distributions having a given equation of state lie between these two extreme cases. The form of the material-energy tensor in case of viscous fluids has recently been given by Lichnerowicz (1955). Also we know that the form of the energy tensor for the electromagnetic distribution is given by

$$E_j^i = - F_{ja} F^{ia} + \tfrac{1}{4} S_j^i F_{ab} F^{ab}, \tag{1·2}$$

where F_{ij} is the skew-symmetric electromagnetic field tensor.

The various solutions in general relativity corresponding to non–empty fields have been obtained by using either of these field equations. But the form of the energy tensor when the matter and the electromagnetic field exist concurrently has not been specified in general relativity. In solving a particular problem of this nature a linear superposition of the two fields T_j^i and E_j^i has recently been considered by Shah and Vaidya (1967). In this paper we examine a more general case of superposition of the perfect fluid distribution and the electromagnetic field within the frame-work of general relativity. For this purpose we take the total energy tensor as the linear sum of the two tensors given in (1·1) and (1·2). The case of discrete particle distribution has also been examined. It is shown in this paper that the field equations thus obtained are consistent. The pressure and density are modified by the presence of the electromagnetic field. Nordstrom solution (2) is given as a particular case as is expected.

Metric and field equations

We consider the spherically symmetric space-time given by

$$ds^2 = -e^{\alpha}\, dt^2 + e^{\beta}\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\,\phi^2). \qquad (2{\cdot}1)$$

where α, β are functions of r and t. The relativistic field equations are

$$R_j{}^i - \tfrac{1}{2}\, R\, \delta_j{}^i = -\, 8\pi\ T_j{}^i, \qquad (2{\cdot}2a)$$

where we take

$$T_j{}^i = T_j{}^i + E_j{}^i. \qquad (2{\cdot}2b)$$

For the metric (2·1) the field equations (2·2a) reduce to

$$-8\pi\,[\,(\varepsilon + p)\, u^1\ u_1 + p + E^1_1\,] = \frac{1}{r_2} - e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right), \qquad (2{\cdot}3a)$$

$$-8\pi\,[\,(\varepsilon + p)\, u^2\, u_2 + p + E^2_2\,] = -\,8\pi\,[\,(\varepsilon + p)\, u^3\, u_3 + p + E^3_3\,]$$

$$= e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1\beta_1 - \alpha_1{}^2}{4} - \frac{\alpha_{11}}{2}\right) \qquad (2{\cdot}3b)$$

$$-8\pi\,[\,(\varepsilon + p)\, u^4\, u_4 + p + E^4_4\,] = \frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r_2}\right), \qquad (2{\cdot}3c)$$

$$-8\pi\,[\,(\varepsilon + p)\, u^1\, u_4 + E^1_4\,]\, e^{\beta} = 8\pi\, e^{\alpha}\ [\,(\varepsilon + p)\, u^4\, u_1 + E^4_1\,] = -\,\frac{\beta_4}{r} \qquad (2{\cdot}3d)$$

$$8\pi\ E^1_2 = 8\pi\ e^{-\beta}\,[\,\frac{F_{23}}{r^2}\, F_{13}\, \operatorname{cosec}^2\theta - e^{-\alpha}\, F_{24}\, F_{14}\,] = -\,8\pi\,(\varepsilon + p)\, u^1\, u_2, \qquad (2{\cdot}3e)$$

$$8\pi\ E^1_3 = 8\pi\, e^{-\beta}\,[\frac{F_{12}}{r^2}\, F_{32} - e^{-\alpha}\, F_{14}\, F_{34}\,] \qquad = -\,8\pi\,(\varepsilon + p)\, u^1\, u_3, \qquad (2{\cdot}3f)$$

$$8\pi\ E^2_3 = \frac{8\pi}{r^2}\,[\, e^{-\beta}\, F_{21}\, F_{31} - e^{-\alpha}\, F_{24}\, F_{34}\,] \qquad = -\,8\pi\,(\varepsilon + p\,)\, u^2\, u_3, \qquad (2{\cdot}3g)$$

$$8\pi\ E^2_4 = \frac{8\pi}{r^2}\,[e^{-\beta}\, F_{21}\ F_{41} + \frac{F_{23}}{r^2}\ F_{43}\, \operatorname{cosec}^2\theta\,] \qquad = -\,8\pi\,(\varepsilon + p)\, u^2\, u_4, \qquad (2{\cdot}3h)$$

$$8\pi\ E^3_4 = \frac{8\pi}{r^2}\,[\ e^{-\beta}\, F_{31}\, F_{41} + \frac{F_{32}}{r^2}\ F_{42} \mid \operatorname{cosec}^2\theta \qquad = -\,8\pi\,(\varepsilon + p)\, u^3\, u_4, \qquad (2{\cdot}3i)$$

where

$$\alpha_1 = \frac{\partial\alpha}{\partial r}\,,\ \alpha_4 = \frac{\partial\alpha}{\partial t}\,,\ \text{etc.}$$

From equations (2·3 $a, \ldots, i$) we have

$$E^3_3 - E^2_2 = (\varepsilon + p)\,(u^2\, u_2 - u^3\, u_3)$$

$$= \frac{1}{r^2} \left[e^{-\beta} (F_{13})^2 - e^{-\alpha} (F_{34})^2 \right] \operatorname{cosec}^2 \theta - \frac{1}{r^2} \left[e^{-\beta} (F_{12})^2 - e^{-\alpha} (F_{24})^2 \right], \qquad (2{\cdot}4)$$

$$(u^1)^2 = (\varepsilon + p)^{-1} e^{2\beta} k, \qquad (2{\cdot}5a)$$

where
$$k = \frac{1}{r^2} \frac{\left[(E^1_2)^2 - (E^1_3)^2 \operatorname{cosec}^2\theta \right]}{E^3_3 - E^2_2}$$

$$(u_2)^2 = (\varepsilon + p)^{-1} \frac{(E^1_2)^2}{k}, \qquad (2{\cdot}5b)$$

$$(u_3)^2 = (\varepsilon + p)^{-1} \frac{(E^1_3)^2}{k}, \qquad (2{\cdot}5c)$$

$$(u_4)^2 = (\varepsilon + p)^{-1} r^4 \frac{(E^2_4)^2}{(E^1_2)^2} k, \qquad (2{\cdot}5d)$$

$$-p = e^{\beta} k + E^1_1 + \frac{1}{8\pi r^2} - \frac{e^{-\beta}}{8\pi} \left(\frac{1}{r^2} + \frac{\alpha_1}{r} \right), \qquad (2{\cdot}6a)$$

$$-\varepsilon = \frac{1}{r^2 k} [E^1_2)^2 + (E^1_3)^2 \operatorname{cosec}^2 \theta] - e^{-\alpha} r^4 \frac{(E^2_4)^2}{(E^1_2)^2} k - E^1_1 - \frac{1}{8\pi r^2} + \frac{e^{-\beta}}{8\pi} \left(\frac{1}{r^2} + \frac{\alpha_1}{r} \right) \qquad (2{\cdot}6b)$$

$$(E^1_3 E^2_4)^2 = (E^1_2 E^3_4)^2 \sin^4\theta, \qquad (2{\cdot}7a)$$

$$(E^1_2 E^1_3)^2 = r^4 (E^2_3)^2 k^2, \qquad (2{\cdot}7b)$$

$$8\pi \left[E^1_1 - E^2_2 + e^{\beta} k - \frac{(E^1_2)^2}{r^2 k} \right] = e^{-\alpha} \left(\frac{\beta_{44}}{2} + \frac{{\beta_4}^2}{4} - \frac{\alpha_4 \beta_4}{4} \right)$$

$$+ e^{-\beta} \left(\frac{\beta_1 + \alpha_1}{2r} + \frac{\alpha_1 \beta_1 - {\alpha_1}^2}{4} - \frac{\alpha_{11}}{2} + \frac{1}{r^2} \right) - \frac{1}{r^2} \qquad (2{\cdot}7c)$$

$$8\pi \left[E^1_1 - E^4_4 + e^{\beta} k + e^{-\alpha} \frac{(E^2_4)^2}{(E^1_2)^2} r^4 k \right] = \frac{e^{-\beta}}{r} (\alpha_1 + \beta_1) - \frac{1}{r^2}, \qquad (2{\cdot}7d)$$

$$8\pi r^2 \frac{E^2_4}{E^1_2} k = \pm \left(8\pi E^1_4 - e^{-\beta} \frac{\beta_4}{r} \right) \qquad (2{\cdot}7e)$$

Thus we have five algebraic equations, *viz.* (2·7*a*, *b*, *c*, *d*, *e*) in eight unknowns *viz.* F_{ij}, α, β. Apart from these algebraic conditions F_{ij}'s have to satisfy Einstein-Maxwell differential equations

$$F_{[ij,\, k]} = 0, \qquad (2{\cdot}8)$$

and

$$F^{ij}_{\ ;j} = J^i, \qquad (2{\cdot}9)$$

where a suffix preceded by a comma (,) stands for partial differentiation and that by a semi-colan (;) indicates covariant differentiation.

The number of algebraic equations is less than the number of unknowns. In the next section we shall investigate the consequences of imposing restrictions on the unknowns.

Radial flow of Matter

We assume that the flow of matter takes place along the radial direction only, *i.e.*

$$u^2 = u^3 = 0.$$

Equations (1·1), (2·3*b*, *e*, *f*. *g*, *h*, *i*) and (2·4) reduce to

$$u^1 u_1 + u^4 u_4 = -1, \tag{3·1}$$

$$-8\pi (p + E^2_2) = -8\pi (p + E^3_3)$$

$$= e^{-\alpha} \left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4 \beta_4}{4} \right) + e^{-\beta} \left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1 \beta_1 - \alpha_1{}^2}{4} - \frac{\alpha_{11}}{2} \right) \tag{3·2a}$$

$$\frac{F_{23}}{r^2} F_{13} \operatorname{cosec}^2\theta - e^{-\alpha} F_{14} F_{24} = 0, \tag{3·2b}$$

$$\frac{F_{12}}{r^2} F_{32} - e^{-\alpha} F_{14} F_{34} = 0, \tag{3·2c}$$

$$e^{-\beta} F_{21} F_{31} - e^{-\alpha} F_{24} F_{34} = 0, \tag{3·2d}$$

$$e^{-\beta} F_{21} F_{41} + \frac{F_{23}}{r^2} F_{43} \operatorname{cosec}^2\theta = 0, \tag{3·2e}$$

$$e^{-\beta} F_{31} F_{41} + \frac{F_{32}}{r^2} F_{42} = 0, \tag{3·2f}$$

$$e^{-\beta} (F_{13})^2 - e^{-\alpha} (F_{34})^2 - [e^{-\beta} (F_{12})^2 - e^{-\alpha} (F_{24})^2] \sin^2\theta = 0. \tag{3·2g}$$

From (3·2*d*) and (3·2*g*) we get

$$\left.\begin{aligned} (F_{13})^2 &= - e^{\beta-\alpha} (F_{24})^2 \sin^2\theta, \\ (F_{12})^2 &= - e^{\beta-\alpha} (F_{34})^2 \sin^2\theta, \end{aligned}\right\} \quad Or \quad \left\{\begin{aligned} (F_{13})^2 &= e^{\beta-\alpha} (F_{34})^2, \\ (F_{12})^2 &= e^{\beta-\alpha} (F_{24})^2 \end{aligned}\right. \tag{3·2h}$$

Since the first set of equations (3·2*h*) gives imaginary values for F_{31} and F_{21} we take the second set of (3·2*h*), *i.e.*

$$F_{12} = \pm e^{\frac{\beta-\alpha}{2}} F_{24}, \tag{3·3a}$$

$$F_{13} = \pm e^{\frac{\beta-\alpha}{2}} F_{34}. \tag{3·3b}$$

In view of (3·3*a*, *b*) the equations (3·2*b*), (3·2*c*), (3·2*e*) and (3·2*f*) reduce to

$$\frac{(F_{23} F_{34})^2}{r^4} \operatorname{cosec}^4\theta - e^{-\alpha-\beta} (F_{14} F_{24})^2 = 0, \tag{3·4a}$$

$$(F_{23} F_{24})^2 r^4 - e^{-\alpha-\beta} (F_{14} F_{34})^2 = 0, \tag{3·4b}$$

and (3·2*d*) reduces to an identity.

Now equations (3·4*a*, *b*) lead to the following cases :

Case (*ia*) $\quad F_{14}, F_{23}, F_{24}, F_{34} \neq 0,$

In this case (3·3*a*) and (3·3*b*) give

$$F_{34} = \pm F_{24} \sin \theta, \tag{3·5a}$$

$$F_{23} = \pm\, r^2 e^{\frac{-\alpha-\beta}{2}} F_{14} \sin\theta, \tag{3·5b}$$

and further we get

$$\left.\begin{aligned} E^1_1 &= e^{-\alpha-\beta} (F_{14})^2 - \frac{2}{r^2}\, e^{-\alpha} (F_{24})^2, \\ E^2_2 &= E^3_3 = -\, e^{-\alpha-\beta} (F_{14})^2, \\ E^4_4 &= e^{-\alpha-\beta} (F_{14})^2 + \frac{2}{r^2}\, e^{-\alpha} (F_{24})^2, \\ E^1_4 &= \pm \frac{2}{r^2}\, e^{\frac{-\alpha-\beta}{2}} (F_{24})^2. \end{aligned}\right\} \tag{3·6}$$

From (3·5*a*, *b*), (3·6), (2·3*a*, *c*, *d*), (3·1) and (3·2*a*) we get

$$-8\pi\, p = -8\pi\, e^{-\alpha-\beta} (F_{14})^2 + e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^{\,2}}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1\beta_1-\alpha_1^{\,2}}{4} - \frac{\alpha_{11}}{2}\right) \tag{3·7a}$$

$$8\pi\, \varepsilon = 3(8\pi)\, e^{-\alpha-\beta} (F_{14})^2 + \frac{2}{r^2} - e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^{\,2}}{4} - \frac{\alpha_4\beta_4}{4}\right) + \\ + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1^{\,2}-\alpha_1\beta_1}{4} + \frac{\alpha_{11}}{2} - \frac{2}{r^2}\right) \tag{3·7b}$$

$$(u^1)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}\, e^{-\beta}\left[-\,8\pi\, p + e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right) - \frac{1}{r^2} + 8\pi\, e^{-\alpha}\left\{\frac{2}{r^2} (F_{24})^2 - e^{-\beta} (F_{14})^2\right\}\right], \tag{3·7c}$$

$$(u^4)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}\, e^{-\alpha}\left[8\pi\, p + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} + 8\pi\left\{\frac{2}{r^2} (F_{24})^2 + e^{-\beta} (F_{14})^2\right\}\right], \tag{3·7d}$$

$$\begin{aligned} (F_{24})^2 = \Bigg[& e^{-\beta}\, \frac{\beta_4^{\,2}}{r^2} - e^{\alpha-2\beta}\, \frac{\alpha_1\beta_1}{r^2} - e^{\alpha}\Bigg\{2(8\pi)\, c^{-\alpha-\beta} (F_{14})^2 - e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^{\,2}}{4} - \frac{\alpha_4\beta_4}{4}\right) - \\ & - e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1\,\beta_1 - \alpha_1^{\,2}}{4} - \frac{\alpha_{11}}{2} + \frac{1}{r^2}\right) + \frac{1}{r^2}\Bigg\}\Bigg\{-\,\frac{1}{r^2} - 2\,(8\pi)\, e^{-\alpha-\beta} (F_{14})^2 \\ & + e^{-a}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^{\,2}}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\alpha_1-\beta_1}{2r} + \frac{\alpha_1\beta_1-\alpha_1^{\,2}}{4} - \frac{\alpha_{11}}{2} + \frac{1}{r^2}\right)\Bigg\}\Bigg] \\ & \div \frac{2}{r^2}\,(8\pi)\left[\frac{e^{-\beta}}{r}\,(\alpha_1+\beta_1) \pm 2\, e^{\frac{-\alpha-\beta}{2}}\, \frac{\beta_4}{r}\right] \end{aligned} \tag{3·8}$$

Thus the algebraic gravitational field equations enable us to express ε, p, u^1, u^4 and F_{24} all in terms of F_{14}, α and β. Now F_{14} will be determined from Maxwell's equations in terms of α, β and when suitable conditions of state are imposed on ε and p, α, β can be determined from (3·7*a*, *b*)

When in (3·3*a*, *b*) and (3·5*a*) positive sign is taken, Maxwell's equations (2·8) reduce to

$$e^{\frac{\beta-\alpha}{2}} (F_{24,3} - F_{34,2}) + F_{23,1} = 0, \tag{3·9a}$$

$$(e^{\frac{\beta-\alpha}{2}} F_{34})_{,4} + F_{34,1} - F_{14,2} \sin\theta = 0, \qquad (3\cdot 9b)$$

$$(e^{\frac{\beta-\alpha}{2}} F_{34})_{,4} + F_{34,1} - F_{14,3} = 0, \qquad (3\cdot 9c)$$

$$-F_{24,3} + F_{34,2} + F_{23,4} = 0, \qquad (3\cdot 9d)$$

From (3·9*d*) and (3·9*c*) we get

$$F_{14,3} - F_{14,2} \sin\theta = 0,$$

which leads to the solution

$$F_{14} \equiv F_{14}\,[\,\{\,\phi - \log(\cot\theta + \operatorname{cosec}\theta)\,\},\, r,\, t\,].$$

Similarly other solutions arising out of different combinations of signs in (3·3*a*, *b*) and (3·5*a*) can be obtained.

Case (*ib*) Distribution of Discrete Particles.

In the case of discrete particle distribution we have

$$p = 0,$$

and (3·7*a*), (3·8) reduce to

$$8\pi\, e^{-\alpha-\beta} (F_{14})^2 = e^{-\alpha} \left(\frac{\beta_{44}}{2} + \frac{\beta_4^{\,2}}{4} - \frac{\alpha_4 \beta_4}{4} \right) + e^{-\beta} \left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1 \beta_1 - \alpha_1^{\,2}}{4} - \frac{\alpha_{11}}{2} \right), \qquad (3\cdot 10)$$

$$(F_{24})^2 = \Big[\, e^{-\beta} \frac{\beta_4^{\,2}}{r^2} - e^{\alpha - 2\beta} \frac{\alpha_1 \beta_1}{r^2} + e^{\alpha} \{\, 8\pi\, e^{-\alpha-\beta} (F_{14})^2 + \frac{1}{r^2} (1 - e^{-\beta}) \,\}^2$$

$$+ e^{\alpha} \{ 8\pi\, e^{-\alpha-\beta} (F_{14})^2 + \frac{1}{r^2} (1 - e^{-\beta}) \} \frac{\beta_1 - \alpha_1)}{r} \Big]$$

$$\div \frac{2}{r^2} (8\pi) \Big[\, e^{-\beta} \frac{(\alpha_1 + \beta_1)}{r} \pm 2\, e^{\frac{-\alpha-\beta}{2}} \frac{\beta_4}{r} \Big]. \qquad (3\cdot 11)$$

From (3·10) and (3·11) we conclude that F_{14} and F_{24} are functions of r and t. So Maxwell's equations will become

$$(\, r^2 e^{\frac{-\alpha-\beta}{2}} F_{14})_{,1} \sin\theta - e^{\frac{\beta-\alpha}{2}} F_{24} \cos\theta = 0, \qquad (3\cdot 12a)$$

$$\pm\, (e^{\frac{\beta-\alpha}{2}} F_{24})_{,4} + F_{24,1} = 0, \qquad (3\cdot 12b)$$

$$(r^2 e^{\frac{-\alpha-\beta}{2}} F_{14})_{,4} \sin\theta + F_{24} \cos\theta = 0. \qquad (3\cdot 12c)$$

Equations (3·12*a*) and (3·12*c*) will hold at all points only when

$$(r^2 e^{\frac{-\alpha-\beta}{2}} F_{14})_{,1} = 0,$$

and

$$F_{24} = 0.$$

But we started with the assumption that none of the F_{ij}'s is zero.

Hence discrete particle distribution does not permit all the F_{ij}'s to be non-zero.

Case (*iia*) $\qquad F_{14} = F_{23} = 0.$

Then

$$\left.\begin{aligned} E_1^1 &= -E_4^4 = -\frac{e^{-\alpha}}{r^2}\left[(F_{24})^2 + (F_{34})^2 \operatorname{cosec}^2\theta\right], \\ E_2^2 &= E_3^3 = 0, \\ E_4^1 &= \pm \frac{e^{\frac{-\alpha-\beta}{2}}}{r^2}\left[(F_{24})^2 + (F_{34})^2 \operatorname{cosec}^2\theta\right]. \end{aligned}\right\} \tag{3·13}$$

Equations (3·1), (3·2*a*), (2·3*a*, *c*, *d*) and (3·13) will give

$$-8\pi p = e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1\beta_1-\alpha_1^2}{4} - \frac{\alpha_{11}}{2}\right), \tag{3·14a}$$

$$8\pi\varepsilon = \frac{2}{r^2}(1-e^{-\beta}) - e^{-\alpha}\left(\frac{\beta_{44}}{4} + \frac{\beta_4^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1^2-\alpha_1\beta_1}{4} + \frac{\alpha_{11}}{2}\right), \tag{3·14b}$$

$$(u^1)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}e^{-\beta}\left[-8\pi p - \frac{1}{r^2} + e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha^1}{r}\right) + \frac{8\pi e^{-\alpha}}{r^2}\{(F_{24})^2 + (F_{34})^2\operatorname{cosec}^2\theta\}\right] \tag{3·14c}$$

$$(u^4)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}e^{-\alpha}\left[+8\pi p + \frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right) + \frac{8\pi e^{-\alpha}}{r^2}\{(F_{24})^2 + (F_{34})^2\operatorname{cosec}^2\theta\}\right] \tag{3·14d}$$

$$\begin{aligned}\frac{8\pi}{r^2}\{(F_{24})^2 + (F_{34})^2\operatorname{cosec}^2\theta\} = {} & \left[e^{-\beta}\frac{\beta_4^2}{r^2} + e^{\alpha}\left\{\frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right) + 8\pi p\right\}\left\{\frac{1}{r^2}\right.\right. \\ & \left.\left. - e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right) + 8\pi p\right\}\right] \div \left[\frac{e^{-\beta}}{r}(\alpha_1+\beta_1) \pm 2e^{\frac{-\alpha-\beta}{2}}\frac{\beta_4}{r}\right]. \end{aligned} \tag{3·15}$$

and equations (2·8) will imply

$$\left.\begin{aligned} F_{24,3} - F_{34,2} &= 0, \\ F_{24,1} \pm \left(e^{\frac{\beta-\alpha}{2}}F_{24}\right)_{,4} &= 0, \\ F_{34,1} \pm \left(e^{\frac{\beta-\alpha}{2}}F_{34}\right)_{,4} &= 0. \end{aligned}\right\} \tag{3·16}$$

Case (*iib*) For discrete particle distribution, *i.e.*, $p = 0$ (3·14*a*) is specified by

$$\alpha = \beta = f(r+t) + g(r-t), \tag{3·17}$$

and (3·14*b*, *c*, *d*) and (3·15) will turn out to be

$$8\pi\varepsilon = \frac{2}{r^2}(1 - e^{-f-g}), \tag{3·18a}$$

$$(u^1)^2 = \frac{e^{-f-g}}{8\pi\varepsilon}\left[-\frac{1}{r^2} + e^{-f-g}\left(\frac{1}{r^2} + \frac{f'+\dot{g}}{r}\right) + \frac{e^{f+g}}{4\dot{g}}\left\{\frac{(1-e^{-f-g})^2}{r^3} - \frac{4}{r}e^{-2(f+g)}f'\dot{g}\right\}\right] \tag{3·18b}$$

$$(u^4)^2 = \frac{e^{-f-g}}{8\pi\varepsilon}\left[\frac{1}{r^2} + e^{-f-g}\left(\frac{f'+\dot{g}}{r} - \frac{1}{r^2}\right) + \frac{e^{f+g}}{4\dot{g}}\left\{\frac{(1-e^{-f-g})^2}{r^3} - \frac{4}{r}\,e^{-2(f+g)}\,f'\dot{g}\right\}\right] \tag{3·18c}$$

$$\frac{8\pi}{r^2}\{(F_{24})^2 + (F_{34})^2 \operatorname{cosec}^2\theta\} \equiv \frac{e^{2(f+g)}}{4f' \text{ (or } 4\dot{g})}\left[\frac{(1-e^{-f-g})^2}{r^3} - \frac{4}{r}\,e^{-2(f+g)}\,f'\,\dot{g}\right] \tag{3·19}$$

where

$$f' \equiv \frac{df}{d(r+t)}, \quad \dot{g} \equiv \frac{dg}{d(r-t)}.$$

Maxwell's equations (3·16) lead to

$$F_{24,3} - F_{34,2} = 0,$$

$$F_{24} \equiv F_{24}\{(r \pm t), \theta, \phi\},$$

$$F_{34} \equiv F_{34}\{(r \pm t), \theta, \phi\},$$

Further if we take $F_{34} = 0$ (3·16) will give

$$F_{24} \equiv F_{24}\{(r \pm t), \theta\}$$

If $F_{24} \equiv F_{24}\{(r+t), \theta\}$ then $\dot{g} = 0$ and right hand side of (3·18*b*, *c*) will become infinite. So we take

$$F_{24} \equiv F_{24}\{(r-t), \theta\},$$

which implies $f' = 0$, *i.e.*, $f =$ constant which can be taken to be zero without loss of generality. And (3·18*a*, *b*, *c*) and (3·19) are

$$8\pi\,\varepsilon = \frac{2}{r^2}(1 - e^{-g}),$$

$$(u^1)^2 = \frac{e^{-g}}{\theta\pi\varepsilon}\left[-\frac{1}{r^2} + e^{-g}\left(\frac{1}{r^2} + \frac{\dot{g}}{r}\right) + \frac{e^{g}}{4\dot{g}}\,\frac{(1-e^{-g})^2}{r^3}\right]$$

$$(u^4)^2 = \frac{e^{-g}}{8\pi\varepsilon}\left[\frac{1}{r^2} + e^{-g}\left(\frac{\dot{g}}{r} - \frac{1}{r^2}\right) + \frac{e^{g}}{4\dot{g}}\,\frac{(1-e^{-g})^2}{r^2}\right],$$

$$8\pi\,(F_{12})^2 = 8\pi\,(F_{24})^2 = \frac{e^{2g}}{4\dot{g}}\,\frac{(1-e^{-g})^2}{r}.$$

In this case the current vector J^i calculated from (2·9) turns out to be zero.

Case (*iiia*) Spherically Symmetric Electromagnetic Distribution.

In case (*i*) and (*ii*) the electromagnetic field has not been assumed to be necessarily spherically symmetric although the material distribution is spherically symmetric. If we impose conditions of spherical symmetry on F_{ij}, from the considerations of groups of motions we find that only two of F_{ij}'s – F_{14}, F_{23} are none-zero. Then

$$\left.\begin{aligned} &F_{12} = F_{13} = 0, \\ &E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = \tfrac{1}{2}\left[e^{-\alpha-\beta}(F_{14})^2 + \frac{(F_{23})^2}{r^4}\operatorname{cosec}^2\theta\right]^2, \\ &E^1_4 = 0. \end{aligned}\right\} \tag{3·20}$$

Equations (3·1), (3·2a), (2·3a, c, d) and (3·20) give

$$-8\pi p = -8\pi E^1_1 + e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\beta_1\alpha_1-\alpha_1^2}{4} - \frac{\alpha_{11}}{2}\right) \quad (3·21a)$$

$$8\pi\varepsilon = 3(8\pi)E^1_1 + \frac{2}{r^2}(1-e^{-\beta}) - e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \frac{\alpha_4\beta_4}{4}\right)$$

$$+ e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1^2-\alpha_1\beta_1}{4} + \frac{\alpha_{11}}{2}\right), \quad (3·21b)$$

$$(u^1)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}\, e^{-\beta}\left[-8\pi p - 8\pi E^1_1 - \frac{1}{r^2} + e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right)\right], \quad (3·21c)$$

$$(u^4)^2 = \frac{(\varepsilon+p)^{-1}}{8\pi}\, e^{-\alpha}\left[8\pi p + 8\pi E^1_1 + \frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right)\right. \quad (3·21d)$$

$$4(8\pi)E^1_1 = -\left\{\frac{e^{-\beta}}{r}(\beta_1-\alpha_1) - \frac{2}{r^2}(e^{-\beta}-1) - 2A\right\} \pm \left[\left\{\frac{e^{-\beta}}{r}(\beta_1-\alpha_1) + \frac{2}{r^2}(1-e^{-\beta}) - 2A\right\}^2\right.$$

$$-4\left\{e^{-\alpha-\beta}\frac{\beta_4^2}{r^2} + e^{\alpha-\beta}\left[\frac{e^{-\beta}}{r}(\alpha_1-\beta_1)\left(A + \frac{e^{-\beta}}{r^2} - \frac{1}{r^2}\right) + \frac{e^{-2\beta}}{r^4}\right.\right.$$

$$\left.\left.\left. -2\frac{e^{-\beta}}{r^4} + \frac{1}{r^4} - e^{-2\beta}\frac{\alpha_1\beta_1}{r^2} + A^2 + 2\frac{A}{r^2}(e^{-\beta}-1)\right]\right\}\right]^{1/2}, \quad (3·22)$$

where

$$A = e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1\beta_1-\alpha_1^2}{4} - \frac{\alpha_{11}}{2}\right).$$

Equations (2·8) imply

$$\left.\begin{aligned} F_{14} &\equiv F_{14}(r, t), \\ F_{23} &\equiv F_{23}(\theta, \phi). \end{aligned}\right\} \quad (3·23)$$

Denoting the right hand side of (3·22) by $B\ (r, t)$, we obtain in view of (3·20)

$$4(8\pi)(F_{23})^2 = r^4\,[2B - 4(8\pi)\,e^{-\alpha-\beta}F_{14})^2\,]\sin^2\theta, \quad (3·24)$$

which by virtue of (3·23) implies

$$r^4\,[2B - 4(8\pi)\,e^{-\alpha-\beta}(F_{14})^2\,] = k_1 \ (\text{constant}),$$

or

$$4(8\pi)(F_{14})^2 = e^{\alpha+\beta}\left((2B - \frac{k_1}{r^4}\right). \quad (3·25)$$

In particular if we assume (3·17) to hold in this case, then (3·21a, b, c, d) and (3·22) will become

$$8\pi p = 8\pi E^1_1 = -\frac{1}{2r^2}(1 - e^{-f-g}) \pm \frac{e^{-f-g}}{r}(f'\,\dot{g})^{1/2}, \quad (3·26a)$$

$$8\pi\,\varepsilon = 3\,(8\pi)\,p + \frac{2}{r^2}\,(1 - e^{-f-g}), \tag{3·26b}$$

$$(u^1)^2 = \frac{(\varepsilon + p)^{-1}}{8\pi}\, e^{-f-g}\left[-\frac{2}{r} e^{-f-g}\,(f'\,\dot{g})^{1/2} + e^{-f-g}\,\frac{(f' + \dot{g})}{r}\right], \tag{3·26c}$$

$$(u^4)^2 = \frac{(\varepsilon + p)^{-1}}{8\pi}\, e^{-f-g}\left[\frac{2}{r}\, e^{-f-g}\,(f'\dot{g})^{1/2} + e^{-f-g}\frac{(f' + \dot{g})}{r}\right], \tag{3·26d}$$

which for $p > 0,\ \varepsilon > 0,\ \varepsilon > 3p,\ (u^1)^2 > 0$ and $(u^4)^2 > 0$ imply

$$e^{f+g} > 0,\ e^{-f-g}\,(f'\dot{g})^{1/2} > \frac{1}{2r}\,(1 - e^{-f-g}), \frac{f' + \dot{}}{2} > (f'\dot{g})^{1/2} \tag{3 27}$$

Case $(iiib)$ In the case of discrete particle distribution (3·21*a*, *b*, *c*, *d*) and (3·22) lead to

$$8\pi\, E^1_1 = e^{-\alpha}\, \left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4\beta_4}{4} + e^{-\beta}\,\left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1\beta_1 - \alpha_1{}^2}{4} - \frac{\alpha_{11}}{2}\right)\right. \tag{3·28a}$$

$$8\pi\,\varepsilon = 2\,(8\pi)\,E^1_1 + \frac{2}{r^2}\,(1 - e^{-\beta}) - \frac{e^{-\beta}}{r}\,(\alpha_1 - \beta_1)\,, \tag{3·28b}$$

$$(u^1)^2 = \frac{e^{-\beta}}{8\pi\,\varepsilon}\left[- 8\pi\, E^1_1 - \frac{1}{r^2} + e^{-\beta}\,\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right)\right], \tag{3·28c}$$

$$(u^4)^2 = \frac{e^{-\alpha}}{8\pi\,\varepsilon}\left[\ 8\pi\, E^1_1 + \frac{1}{r^2} + e^{-\beta}\,\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right)\right], \tag{3·28d}$$

$$\left[8\pi\, E^1_1 + \frac{1}{r^2} - e^{-\beta}\,\left(\frac{\alpha_1}{r} + \frac{1}{r^2}\right)\right]\left[8\pi\, E^1_1 + \frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right) = - e^{-\alpha-\beta}\,\frac{\beta_4{}^2}{r^2}\right., \tag{3·29}$$

and

$$8\pi\,(F_{23})^2 = k_2\,\sin^2\theta, \tag{3·30}$$

where

$$k_2 = r^4\left[2\, e^{-\alpha}\,\left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + 2\, e^{-\beta}\left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1\beta_1 - \alpha_1{}^2}{4} - \frac{\alpha_{11}}{2}\right) - 8\pi\, e^{-\alpha-\beta}\,(F_{14})^2\right],$$

or

$$8\pi\,(F_{14})^2 = e^{\alpha+\beta}\left[\ 2e^{-\alpha}\,\left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + 2\, e^{-\beta}\,\left(\frac{\beta_1 - \alpha_1}{2r} + \frac{\alpha_1\beta_1 - \alpha_1{}^2}{4} - \frac{\alpha_{11}}{2}\right) - \frac{k_2}{r^4}\right]. \tag{3·31}$$

Further in the absence of matter we have

$$\varepsilon = p = 0,$$

and equations (2·3*a*, *b*, *c*, *d*) are

$$\left.\begin{aligned} -8\pi E_1^1 &= \frac{1}{r^2} - e^{-\beta}\left(\frac{1}{r^2} + \frac{\alpha_1}{r}\right), \\ 8\pi E_1^1 &= e^{-\alpha}\left(\frac{\beta_{44}}{2} + \frac{\beta_4{}^2}{4} - \frac{\alpha_4\beta_4}{4}\right) + e^{-\beta}\left(\frac{\beta_1-\alpha_1}{2r} + \frac{\alpha_1\beta_1-\alpha_1{}^2}{4} - \frac{\alpha_{11}}{2}\right], \\ -8\pi E_{\mathrm{I}}^{\mathrm{T}} &= \frac{1}{r^2} + e^{-\beta}\left(\frac{\beta_1}{r} - \frac{1}{r^2}\right), \\ \beta_4 &= 0. \end{aligned}\right\} \quad (3\cdot32)$$

Equations (3·32) will lead to the solution

$$e^{\beta} = e^{-\alpha} = \left(1 - \frac{a}{r} - \frac{b}{r^2}\right)^{-1}.$$

and

$$4\pi\left[\,(F_{14})^2 + \frac{(F_{23})^2}{r^4}\operatorname{cosec}^2\theta\,\right] = \frac{b}{r^4}.$$

If we take here $F_{23} = 0$, Nordström solution is obtained as a particular case.

References

1. Lichnerowicz. A. Theories Relativistes De La Gravitation Et de L' Electro-Magnetisme, Masson Et C^{ie}, Editeurs, 93-95, 1955.
2. Tolman, R. C. Relativity Thermodynamics and Cosmology, *Clarenden Press, Oxford,* 335.
3. Shah, Y. P. and Vaidya, P. C. *Ann Inst. Henri Poincare,* **6**(3), 219–224, 1967.

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On the Rate of Decay of Solutions of Neutral Functional Differential Systems

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1. In recent years differential systems involving rather arbitrary retarded arguments have been studied more and more intensively. This is due to the importance of such systems in various branches of technology. Recently, Driver [1, 2], Lakshmikantham and Lella [4] and others have dealt with the neutral functional differential systems, and have obtained several remarkable results. However, all the theory developed concerns itself with the stability and boundedness problems of neutral functional differential systems by considering one-sided estimates of solutions. In many cases of practical applications, it is natural to expect that, an estimation of the lower bound for the rate at which the solutions approach the origin or the invariant set, would yield useful refinements of stability notions. In the present paper upper and lower bounds are determined for the rate of growth or decay of solutions of the neutral functional differential systems which arise in one approach in the two-body problem of classical electrodynamics. These bounds are obtained by using comparison principle and the extension of Lyapunov's method. We introduce the concepts of relative-strict-stability and relative-strict-boundedness of the neutral functional differential systems and the two-sided estimates obtained ensure that the relative motion remains in tube-like domains

2. Consider the two systems of neutral functional differential equations

$$x'(t) = f(t, x(t), x(g(t,x(t))), x'(g(t,x(t)))),\ t > t_0 \tag{2·1}$$

with $x(t) = \phi(t)$ on $[\alpha, t_0]$,

$$y'(t) = h(t, y(t), y(g(t, y(t))), y'(g(t, y(t)))),\quad t > t_0 \tag{2·2}$$

with $y(t) = \psi(t)$ on $[\alpha, t_0]$,

where $x = (x_1, x_2, \ldots, x_n)$; $y = (y_1, y_2, \ldots, y_n)$; $f = (f_1, f_2, \ldots, f_n)$; $h = (h_1, h_2, \ldots, h_n)$ and $g = (g_1, g_2, \ldots, g_m)$. For $\alpha < t_0$, $\alpha \leqslant g_j(t,x(t)) < t$ and $\alpha \leqslant g_j(t,y(t)) < t$, $j = 1, 2, \ldots, m$, for all $t \geqslant t_0$ and $x'(g_j(t,x(t)))$ and $y'\, g_j(t,y(t)))$ stands for $\frac{dx}{dt}$ and $\frac{dy}{dt}$ evaluated at $g_j(t,x(t))$ and $g_j(t,y(t))$ respectively. The word 'Neutral' is used because the equations could be considered both for retarded arguments as well as for advanced arguments. However, in these systems the retarded arguments will depend upon the dependent variables as well as the independent variable.

Suppose f, h and g are continuous functions of their arguments in their appropriate domains that is f is continuous in t, $x(t)$, $x(g(t,x(t)))$, $x'(g(t,x(t)))$ in the domain $D = D^{1+n+2nm}$ (an open connected set) in $E^{1+n+2nm}$, an $(1+n+2nm)$ dimensional Euclidean space. A function $x(t)$ is said to be a solution of (2·1) if it satisfies the following conditions :

(*i*) $(t, x(t), x(g(t, x(t)))) \in D - G$

for each compact set $G \subset D$, $t_0 \leqslant t < \beta$,

(*ii*) $x(t) = \phi\,(t)$, $\alpha \leqslant t \leqslant t_0$,

(*iii*) $x'(t) = f(t, x(t), x(g(t, x(t))), x'\,(g(t,x(t))))$

for almost all $t \in (t_0, \beta)$.

Similarly we can define $y(t)$ as the solution of (2·2).

The existence and uniqueness of Neutral Functional–Differential Equations have been discussed by many authors. Driver [1, 2] has given references of many results obtained by various authors in this direction.

Let $x(t)$ and $y(t)$ be any two solutions of (2·1) and (2·2) respectively. Let $\sigma\,[x,y] = \| x - y \|$ denote the distance between two points x and y. Our interest lies in the following definitions. The systems (2·1) and (2·2) are said to be

(S_1) *relatively–strictly–equi–stable*, if for each $\epsilon_1 > o$, $t_0 \geqslant o$, it is possible to find positive functions $d_1 = d_1\ (t_0, \epsilon_1)$, $d_2 = d_2\,(t_0, \epsilon_1)$ and $\epsilon_2 = \epsilon_2\ (t_0, \epsilon_1)$ which are continuous in t_0 for each ϵ_1, $\epsilon_2 < d_2 \leqslant d_1 < \epsilon_1$ such that

$$\epsilon_2 < \sigma\,[x(t), y(t)] < \epsilon_1,\ t > t_0,$$

whenever

$$d_2 \leqslant \sigma\,[\,\phi\,(t),\ \psi(t)\,] \leqslant d_1,\ \alpha \leqslant t \leqslant t_0\ ;$$

(S_2) *relatively-strictly-equi-norm-bounded*, if for each $\alpha_1 > o$, $t_o \geqslant o$, it is possible to find for every α_2 satisfying $o < \alpha_2 < \alpha_1$, positive functions $\beta_1 = \beta_1\,(t_0, \alpha_1)$, $\beta_2 = \beta_2\,(t_0, \alpha_1, \alpha_2)$, which are continuous in t_0 for each α_1, $\beta_2 < \beta_1$, $\beta_2 < \alpha_2$ such that

$$\beta_2 < \sigma\ [\,x(t), y(t)\,] < \beta_1,\ t > t_0,$$

whenever

$$\alpha_2 \leqslant \sigma\ [\,\phi\,(t),\ \psi\,(t)\,] \leqslant \alpha_1,\quad \alpha \leqslant t \leqslant t_o.$$

3. Let $V(t,x,y) \geqslant o$ be defined and continuous function of its argument in its appropriate domain, and satisfies a Lipschitz condition in x and y for each t. Define, for small $\lambda > o$

$$(3\cdot1)\quad V^*(t,x,y) = \limsup_{\lambda\to 0^+} \tfrac{1}{\lambda}\big[\ V(t+\lambda,\ x(t)+\lambda\ f(t,x(t),\ x(g(t,x(t))),\ x'(g(t,x(t))))),$$
$$y(t)+\lambda\ h(t,y(t),y(g(t,y(t))),y'(g(t,y(t)))))) - V(t,x(t),y(t))\big].$$

We formulate the following lemma which is useful for our subsequent discussion.

Lemma. *Let the function* $V^*(t,x,y)$ *of* (3·1) *satisfy the inequality*

$$(3\cdot2)\qquad W_2(t,\ V(t,x,y)) \leqslant V^*(t,x,y) \leqslant W_1(t,\ V(t,x,y))$$

where $W_1(t,r)$ *and* $W_2(t,r)$ *are continuous real valued functions for* $t \varepsilon I = [\,0,\infty),\ r \geqslant 0.$ *Let* $r(t)$ *and* $\rho\ (t)$ *be the maximal and minimal solutions of the scalar differential equations*

$$(3\cdot3)\qquad r' = W_1(t,r),\ r(t_0) = r_0,$$

$$(3\cdot4)\qquad \rho' = W_2(t,\rho),\ \rho(t_0) = \rho_0,$$

respectively for all $t \geqslant t_0$. *If* $x(t)$ *and* $y(t)$ *be any two solutions of* (2·1) *and* (2·2) *with their initial functions* $\phi\ (t)$ *and* $\psi\ (t)$ *respectively on* $[\,a,\ t_0\,]$ *such that*

$$\rho_0 \leqslant V(t_0,\ \phi(t),\ \psi\ (t)) \leqslant r_0,\ a \leqslant t \leqslant t_0,$$

then

$$(3\cdot5)\qquad \rho\ (t) \leqslant V(t,x(t),y(t)) \leqslant r(t),\ t > t_0.$$

Proof. We shall prove the right half of the inequality (3·5), similar reasoning can be used for the left half. Define $m(t) = V(t,x(t)\ y(t))$. then $m(t_0) \leqslant r_0$. Further for small $\lambda > 0$,

$$m(t+\lambda) - m(t) \leqslant C\,[\ \|\ x(t+\lambda) - x(t) - \lambda\ f(t,\ x(t),\ x(g(t,\ x(t))),\ x'(g(t,\ x(t))))\ \|$$
$$+ \|\,y\ (t+\lambda) - y\ (t) - \lambda h(t,y(t),y(g(t,y(t))),y'(g(t\ y(t))))\ \|\ ]$$
$$+\ V(t+\lambda,\ x(t) + \lambda f(t,x\ (t),\ x(g(t,\ x(t))),x'\ (g(t,\ x(t)))),$$
$$y(t)+\lambda h\ (t,y(t),y(g(t,y(t))),y'(g(t,y(t))))) - V(t,\ x(t),y(t))$$

where $C > 0$ is a Lipschitz constant. This together with (3·1) and (3·2) implies the inequality

$$\limsup_{\lambda\to 0^+} \frac{1}{\lambda}\ [\,m(t+\lambda) - m(t)\,] \leqslant W_1\,(t,\ m(t)).$$

The standard argument used in [2] can now be followed to establish the desired right-half of the result in (3·5).

Corresponding to the definition (S_1), we can formulate the definition $(S_1{}^*)$ with respect to (3·3) and (3·4).

$(S_1{}^*)$ For each $\eta_1 > 0,\ t_0 \geqslant 0$, there exist positive functions $\delta_1 = \delta_1(t_0,\ \eta_1)$, $\delta_2 = \delta_2\ (t_0,\ \eta_1)$ and $\eta_2 = \eta_2(t_0,\ \eta_1)$ which are continuous in t_0 for each η_1, $\eta_2 < \delta_2 \leqslant \delta_1 < \eta_1$ such that

$$\eta_2 < \rho\ (t) \leqslant r(t) < \eta_1,\ t > t_0$$

whenever

$$\delta_2 \leqslant \rho_0 \leqslant r_0 \leqslant \delta_1.$$

Definition $(S_2{}^*)$ may be formulated similarly.

4. We list below certain assumptions which will be used subsequently.

$$b(\sigma\,[x,y]) \leqslant V(t,x,y) \leqslant a(\,\sigma\,[x,y]), \tag{4·1}$$

where $a(r)$, $b(r)$ are continuous, strictly increasing functions for $r \geqslant o$ with $a(o) = b(o) = o$,

$$b(r) \to \infty \text{ as } r \to \infty. \tag{4·2}$$

The following theorems provide sufficient conditions for relative-strict-stability and relative-strict-boundedness of the systems (2·1) and (2·2).

Theorem 1. *Let the assumptions of the lemma hold, together with* (4·1). *Then* $(S_1{}^*)$ *implies that the systems* (2·1) *and* (2·2) *are relatively-strictly-equi-stable.*

Proof. Let $\varepsilon_1 > o$ and $t_0 \geqslant o$ be given. Since $(S_1{}^*)$ holds, given $\eta_1 = b(\varepsilon_1) > o$, $t_0 \geqslant o$, there exist positive functions $\delta_1 = \delta_1\ (t_0,\ \eta_1)$, $\delta_2 = \delta_2\ (t_0,\ \eta_1)$ and $\eta_2 = \eta_2\ (t_0,\ \eta_1)$, $\eta_2 < \delta_2 \leqslant \delta_1 < \eta_1$,

such that

$$\eta_2 < \rho(t) \leqslant r(t) \leqslant \eta_1,\ t > t_0, \tag{4·3}$$

whenever

$$\delta_2 \leqslant \rho_0 \leqslant r_0 \leqslant \delta_1. \tag{4·4}$$

Choose an $\varepsilon_2 > o$ such that $a(\varepsilon_2) \leqslant \eta_2$ and $\varepsilon_2 < \varepsilon_1$. Let $x(t)$ and $y(t)$ be any two solutions of (2·1) and (2·2) respectively, such that,

$$\rho_0 \leqslant V(t_0,\ \phi(t),\ \psi(t)) \leqslant r_0, \text{ for } \alpha \leqslant t \leqslant t_0. \tag{4·5}$$

Then it follows from the lemma that

$$\rho\ (t) \leqslant V\ (t,x(t),y(t)) \leqslant r(t), \quad t > t_0. \tag{4·6}$$

Further (4·1), (4·5) and (4·4), show that there exist two positive functions $d_1 = d_1\ (t_0,\ \varepsilon_1)$, $d_2 = d_2\ (t_0,\ \varepsilon_1)$ such that

$$d_2 \leqslant \sigma[\ \phi\ (t),\ \psi\ (t)\ ] \leqslant d_1 \text{ implies } \delta_2 \leqslant V(t_0,\ \phi\ (t),\ \psi\ (t)) \leqslant \delta_1$$

for $\alpha \leqslant t \leqslant t_0$, and vice versa. Thus whenever $d_2 \leqslant \sigma\ [\ \phi\ (t),\ \psi\ (t)\ ] \leqslant d_1$, it follows from the assumptions of the lemma that (4·6) is true. Now we claim that $\varepsilon_2 < \sigma[\ x(t), y\ (t)\ ] < \varepsilon_1$, for $t > t_0$, provided $d_2 \leqslant \sigma\ [\ \phi\ (t),\ \psi\ (t)\ ] \leqslant d_1$, for $\alpha \leqslant t \leqslant t_0$.

Suppose on the contrary, there exist solutions $x(t)$ and $y(t)$ of (2·1) and (2·2) respectively, satisfying $d_2 \leqslant \sigma\ [\ \phi\ (t),\ \ \psi\ (t)\ ] \leqslant d_1$, such that for some $t = t_1 > t_0$,

$\sigma[x(t_1), y(t_1)] = \varepsilon_1$ or $\sigma[x(t_1), y(t_1)] = \varepsilon_2$. In the first instance, using the inequalities in (4·1), (4·6) and (4·3) we arrive at the contradiction

$$b(\varepsilon_1) \leqslant V(t_1, x(t_1), y(t_1)) \leqslant r(t_1) < b(\varepsilon_1).$$

On the other hand, if $\sigma[x(t_1), y(t_1)] = \varepsilon_2$, we arrive to a similar contradiction

$$a(\varepsilon_2) \geqslant V(t_1, x(t_1), y(t_1)) \geqslant \rho(t_1) > \eta_2 \geqslant a(\varepsilon_2),$$

because of the inequalities in (4·1), (4·6) and (4·3). Which proves that the condition (S_1^*) implies (S_1).

THEOREM 2. *Let the assumptions of the lemma hold, together with* (4·1) *and* (4·2). *Then* (S_2^*) *implies that the systems* (2·1) *and* (2·2) *are relatively-strictly-equi-norm-bounded.*

PROOF. Let $\alpha_1 > 0$ and $t_0 \geqslant 0$ be given. Choose α_2 such that $0 < \alpha_2 \leqslant \alpha_1$. Let $\phi(t)$ and $\psi(t)$ be such that $\alpha_2 \leqslant \sigma[\phi(t), \psi(t)] \leqslant \alpha_1$, for $\alpha \leqslant t \leqslant t_0$. Then because of (4·1) it is possible to find positive numbers $\hat{\alpha}_1 = \hat{\alpha}_1(\alpha_1)$ and $\hat{\alpha}_2 = \hat{\alpha}_2(\alpha_2)$ such that

$$\hat{\alpha}_2 \leqslant V(t_0, \phi(t), \psi(t)) \leqslant \hat{\alpha}_1, \ \alpha \leqslant t \leqslant t_0. \tag{4·7}$$

Let (S_2^*) holds. Then given $\hat{\alpha}_1 > 0$, $t_0 \geqslant 0$, there exist, for every $\hat{\alpha}_2$, $0 < \hat{\alpha}_2 \leqslant \hat{\alpha}_1$, two positive functions $l_1 = l_1(t_0, \hat{\alpha}_1)$, $l_2 = l_2(t_0, \hat{\alpha}_1, \hat{\alpha}_2)$, $l_2 < l_1$, $l_2 < \hat{\alpha}_2$ such that

$$l_2 < \rho(t) \leqslant r(t) < l_1, \tag{4·8}$$

whenever

$$\hat{\alpha}_2 \leqslant \rho_0 \leqslant r_0 \leqslant \hat{\alpha}_1. \tag{4·9}$$

Since $b(r) \to \infty$ as $r \to \infty$, it is possible to find positive functions $\beta_1 = \beta_1(t_0, \alpha_1)$, $\beta_2 = \beta_2(t_0, \alpha_1, \alpha_2)$, such that

$$l_1 \leqslant b(\beta_1), \ l_2 \geqslant a(\beta_2), \ \beta_2 < \alpha_2, \ \beta_2 < \beta_1.$$

Now supposing the contrary and proceeding as in the proof of Theorem 1, one can prove that (S_2) is true.

5. Corresponding to the systems (2·1) and (2·2), we consider the systems

$$\begin{aligned} x'(t) &= f(t, x(t), x(g(t,x(t))), x'(g(t, x(t)))) \\ &\quad + F(t, x(t), x(g(t,x(t))), x'(g(t, x(t)))) \\ &\text{with } x(t) = \phi(t) \text{ on } [\alpha, t_0], \end{aligned} \tag{5·1}$$

$$\begin{aligned} y'(t) &= h(t, y(t), y(g(t,y(t))), y'(g(t,y(t)))) \\ &\quad + H(t, y(t) \cdot y(g(t, y(t))), y'(g(t, y(t)))) \\ &\text{with } y(t) = \psi(t) \text{ on } [\alpha, t_0], \end{aligned} \tag{5·2}$$

respectively, where F and H are perturbed functions and obtain similar properties under constantly acting perturbations.

Assume that

$$(5{\cdot}3)\qquad \| F(t, x(t), x(g(t, x(t))), x'(g(t, x(t)))) \| + \| H(t, y(t), y(g(t, y(t))), y'(g(t, y(t)))) \| \leqslant n\, V(t,x,y), \quad (n > o).$$

If the solutions of the systems (5·1) and (5·2) satisfy the definitions (S_1) and (S_2), whenever (5·3) is satisfied, we shall say that the systems (2·1) and (2·2) satisfy the definitions (S_1) and (S_2) weakly. Then it is easy to prove analogous results for weak-relative-strict-stability and weak-relative-strict-boundedness. We merely state the following two theorems.

THEOREM 3. *Let the assumptions of the lemma hold except that the condition* (3·2) *is replaced by*

$$(5{\cdot}4)\qquad W_2(t, V(t,x,y)) \leqslant V^*(t,x,y) + \alpha\, V(t,x,y) \leqslant W_1(t, V(t,x,y))$$

where $\alpha = nC$. *Assume that* (4·1) *holds. If the condition* $(S_1{}^*)$ *holds, then the systems* (2·1) *and* (2·2) *are weakly-relatively-strictly-equi-stable.*

THEOREM 4. *Let the assumptions of the lemma hold except that the condition* (3·2) *is replaced by* (5·4). *Assume that* (4·1) *and* (4·2) *hold. If the condition* $(S_2{}^*)$ *holds, then the systems* (2·1) *and* (2·2) *are weakly-relatively-strictly-equi-norm-bounded.*

We note that many properties including the stability and boundedness studied in [4] satisfied by a scalar differential equation imply the corresponding properties satisfied by the neutral functional differential systems.

References

1. Driver, R. D. Existence and continuous dependence of solutions of a Neutral functional differential equation. *Arch. Rat. Mech. Anal.* **19**, 149–166, 1965.
2. Driver, R. D. Existence and stability of solutions of a Delay Differential Systems. *Arch. Rat. Mech. Anal.* **10**, 401–426, 1962.
3. Hale, J. K. Periodic and almost periodic solutions of functional differential equations. *Arch. Rat. Mech. Anal.*, **15**, 289–304, 1964.
4. Lakshmikantham, V. and Lella, S. Differential and Integral Inequalities, Vol. I, II, *Academic Press, New York and London,* 1969.
5. Pachpatte, B. G. On some n-th order finite difference inequalities. *Proc. Nat. Acad. Sci., India,* **40**, 235 240, 1970.
6. Zubov, V. I. Methods of A. M. Lyapunov and their Applications. *Noordhoff, Amsterdam,* 1964.

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On order of partial sum of Laplace series

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1. Let $f(\theta, \phi)$ be a function defined for the range $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \phi \leqslant 2\pi$, and integrable in the sense of Labesgue on the surface of the sphere S. The Laplace series corresponding to this function is

$$f(\theta, \phi) \sim \frac{1}{2\pi} \Sigma (n + \tfrac{1}{2}) \int_S\!\!\int f(\theta', \phi')\, P_n(\cos \omega)\, d\sigma', \tag{1·1}$$

where

$$\cos \omega = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi'),$$

and

$$d\sigma' = \sin \theta'\, d\sigma'\, d\phi',$$

and the Legendre polynomials $P_n(x)$ is defined by the relation

$$(1 - 2\,xt + t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x). \tag{1·2}$$

We define on the lines of Kogbetliantz[3], by $f(\omega)$ the function,

$$f(\omega) = \frac{1}{2\pi \sin \omega} \int_{c_\omega} f(\theta', \phi')\, d\sigma' , \tag{1·3}$$

where the integral is taken along the small circle c_ω whose centre is the point (θ, ϕ) on the surface of the sphere S and whose curvilinear radius is ω. The series (1·1) then reduces to the form

$$\Sigma (n + \tfrac{1}{2}) \int_0^\pi f(\omega) \sin \omega\, P_n(\cos \omega)\, d\omega. \tag{1·4}$$

We write in this paper

$$F(\omega) = f(\omega)\, (\sin \omega)^{\frac{1}{2} - k}, \tag{1·5}$$

and denote by S_n the nth partial sum of the series (1·1). We also write

$$\left\{ \frac{d}{dx} [P_n(x)] + \frac{d}{dx} [P_{n+1}(x)] \right\}_{x = \cos \omega} = P_n'(\cos \omega) + P'_{n+1}(\cos \omega). \tag{1·6}$$

2. For Fourier series of a function $F(x)$ integrable in the interval $(-\pi, \pi)$ and periodic with period 2π, Szàsz[5] has established the following theorem on the order of partial sums.

Theorem :

(2·1) $$\text{If} \int_0^h |F(x)|\,dx = 0\left(\frac{h}{\log\frac{1}{t}}\right),$$

as $h \to 0$,

then $$\sum_{m=0}^{n} A_m \cos m\,x = 0\,(\log\log n),$$

where $$A_n = \frac{1}{\pi}\int_0^{2\pi} F(x)\cos n\,x\,d\,x.$$

The order of partial sums of Legendre series has been calculated by Wilson[6] who proved the following

Theorem.

(2·2) For $0 \leqslant k \leqslant \frac{1}{2}$,

if the integral
$$\int_0^{\pi} f(\cos\theta)\,(\sin\theta)^{k+\frac{1}{2}}\,d\theta$$

exists, then
$$S_n(\cos\theta) \equiv \sum_{r=0}^{n} a_\nu\,P_\nu(\cos\theta) = \left[\begin{array}{l} 0(n),\ (0 < k \leqslant \frac{1}{2}) \\ 0(\log n),\ (k = 0) \end{array}\right.$$

Later on in 1966 Beohar[1] has proved on the order of partial sum of Legende series the following theorem :

(2·3) *Theorem* : At any interior point x of the interval $(-1, 1)$ for which
$$\Phi(t) \equiv \int_0^t |\phi(u)|\,du = 0\left\{\frac{t}{(\log\frac{1}{t})^{\alpha}}\right\},$$

as $t \to 0$,

then
$$S_n(x) - f(x) = \left[\begin{array}{ll} 0(\log n)^{1-\alpha} & ,\ (0 \leqslant \alpha < 1) \\ 0(\log\log n) & ,\ (\alpha = 1) \end{array}\right.$$

Here we prove the following theorem on the order of partial sum of Laplace series.

(2·4) *Theorem.* If $F(\omega)\ \varepsilon\ \mathrm{Lip}^*\left(\frac{1}{2} - k\right)$,

then
$$S_n = \left[\begin{array}{ll} 0(n^k) & ,\ (0 < k < \frac{1}{2}) \\ 0(\log n) & ,\ (k = 0) \end{array}\right.$$

3. In order to prove the theorem we need the following lemmas :

Lemma 1.[2]

(3·1) $$P_n'(\cos\omega) + P'_{n+1}(\cos\omega) = O(n^2).$$

for all values of ω.

Lemma 2.[2]

(3·2) $$\{P_n'(\cos\omega) + P'_{n+1}(\cos\omega)\}\sin\omega = O(n\sin\omega)$$

in the range $\pi - n^{-1} \leqslant \omega \leqslant \pi$.

Lemma 3.[2]

(3·3) $$[P_n'(\cos\omega) + P'_{n+1}(\cos\omega)]\sin\omega = -\frac{2}{\pi}$$

$$R\,[(n+1)\,e^{i[(n+1)\,\omega + \frac{\pi}{4}]}\,(\cot\tfrac{\omega}{2})^{\frac{1}{2}}\,\{\Gamma\tfrac{1}{2}(n+1)^{-\frac{1}{2}} + O(n^{-3/2}(\sin\omega)^{-1}\}$$

$$+\,O(n^{-\frac{1}{2}}(\sin\omega)^{-\frac{1}{2}}) + O(n^{-\frac{1}{2}}\,\omega^{-1}(\sin\omega)^{-\frac{1}{2}}) + O(n^{-3/2}(\sin\omega)^{-3/2})]$$

in $\alpha_n \leqslant \omega \leqslant \pi - n^{-1}$.

4 Proof of the theorem. From the definition we have

(4·1) $$S_n = \sum_{k=0}^{n}(k+\tfrac{1}{2})\int_0^{\pi} f(\omega)\,P_k(\cos\omega)\sin\omega\,d\,\omega.$$

It is know that [4, pp. 178]

(4·2) $$\sum_{k=0}^{n}(2k+1)\,P_n(x) = \frac{d}{dx}\,[P_n(x)] + \frac{d}{dx}\,[P_{n+1}(x)].$$

Hence

(4·3) $$\sum_{k=0}^{n}(2k+1)\,P_n(\cos\omega = \left\{\frac{d}{dx}\,[P_n(x)] + \frac{d}{dx}[P_{n+1}(x)]\right\}_{x=\cos\omega},$$

$$= P_n'(\cos\omega) + P'_{n+1}(\cos\omega).$$

In view of (1·5), (4·1) and (4·3) S_n is given by

(4·4) $$S_n = \tfrac{1}{2}\int_0^{\pi} F(\omega)\,[P_n'(\cos\omega) + P'_{n+1}(\cos\omega)]\,(\sin\omega)^{k+\frac{1}{2}}\,d\omega,$$

$$= \tfrac{1}{2}\left(\int_0^{\alpha_n} + \int_{\alpha_n}^{\pi-n^{-1}} + \int_{\pi-n^{-1}}^{\pi}\right) F(\omega)\,[P_n'(\cos\omega) + P'_{n+1}(\cos\omega)]\,(\sin\omega)^{k+\frac{1}{2}}\,d\omega,$$

$$= I_1 + I_2 + I_3,$$

where

$$\alpha_n = \begin{cases} n^{k/2-1}, & (0 < k < \frac{1}{2}) \\ (\log n)^{\frac{1}{2}}\,n^{-1}, & (k = 0) \end{cases}$$

(4·5) First, $$I_1 = O(n^2)\int_0^{\alpha_n} 0(\omega^{\frac{1}{2}-k})\,\omega^{k+\frac{1}{2}}\,d\omega,$$

$$= \begin{cases} 0(n^k), & (0 < k < \frac{1}{2}) \\ 0(\log n), & (k = 0) \end{cases}$$

by the use of Lemma 1 and the fact that

$$F(\omega) = 0(\omega^{\frac{1}{2}-k})$$

(4·6) Next,

$$I_3 = O(n) \int_{k-n^{-1}}^{\pi} (\sin \omega)^{k+\frac{1}{2}} \, d\omega,$$

$= O(1)$ by the use of Lemma 2 and the boundedness of $F(\omega)$.

Finally, by the use of Lemma 3 we have

$$(4{\cdot}7) \qquad I_2 = -\frac{1}{\pi} R \left[\int_{a_n}^{\pi-n^{-1}} (n+1)\, e^{i[(n+1)\,\omega+\pi/4]} F(\omega) (\sin \omega)^{k-\frac{1}{2}} (\cot \frac{\omega}{2})^{\frac{1}{2}} \right.$$

$$\{ \Gamma\tfrac{1}{2} (n+1)^{-1/2} + O(n^{-3/4} (\sin \omega)^{-1})\}] \, d\omega$$

$$+ (n^{-1/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega) (\sin \omega)^{k-1} \, d\omega + O(n^{-1/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega)\, \omega^{-1} (\sin \omega)^{k-1} \, d\omega$$

$$+ (n^{-3/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega) (\sin \omega)^{k-2} \, d\omega,$$

$$= I_{2\cdot 1} + I_{2\cdot 2} + I_{2\cdot 3} + I_{2\cdot 4} + I_{2\cdot 5} \,.$$

$$(4{\cdot}7{\cdot}1) \qquad I_{2\cdot 2} = O(n^{-1/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega) (\cot \frac{\omega}{2})^{1/2} (\sin \omega)^{k-3/2} \, d\omega,$$

$$= O(n^{-1/2}) \int_{a_n}^{\pi/2} \omega^{\frac{1}{2}-k} . \, \omega^{-\frac{1}{2}} \, \omega^{k-3/2} \, d\omega + O(n^{-1/2}) \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{k.2} \, d\omega,$$

$$= O(1).$$

$$(4{\cdot}7{\cdot}2) \qquad I_{2\cdot 3} = O(n^{-1/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega) (\sin \omega)^{k-1} \, d\omega,$$

$$= O(n^{-1/2}) \int_{a_n}^{\pi/2} \omega^{\frac{1}{2}-k} \, \omega^{k-1} \, d\omega + O(n^{-\frac{1}{2}}) \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{k-1} \, d\omega,$$

$$= O(1).$$

$$(4{\cdot}7{\cdot}3) \qquad I_{2\cdot 4} = O(n^{-\frac{1}{2}}) \int_{a_n}^{\pi-n^{-1}} F(\omega)\, \omega^{-1} (\sin \omega)^{k-1} \, d\omega,$$

$$= O(n^{-\frac{1}{2}}) \int_{a_n}^{\pi/2} \omega^{\frac{1}{2}-k} \, \omega^{-1} \, \omega^{k-1} \, d\omega + O(n^{-\frac{1}{2}}) \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{k-1} \, d\omega,$$

$$= O(1).$$

$$(4{\cdot}7{\cdot}4) \qquad I_{2\cdot 5} = O(n^{-3/2}) \int_{a_n}^{\pi-n^{-1}} F(\omega) (\sin \omega)^{k-2} \, d\omega,$$

$$= O(n^{-3/2}) \int_{a_n}^{\pi/2} \omega^{\frac{1}{2}-k} \, \omega^{k-2} \, d\omega + O(n^{-3/2}) \int_{\pi/2}^{\pi-n^{-1}} (\sin \omega)^{k-2} \, d\omega,$$

$$= O(1).$$

$$(4{\cdot}7\ 5)\qquad I_{2\cdot 1} = R\left[\frac{1}{\sqrt{\pi}}\ e^{i\pi/4}\,(n+1)^{\frac{1}{2}}\int_{\alpha_n}^{\pi\ n^{-1}} e^{i(n+1)\omega}\,F(\omega)\,(\cot\tfrac{\omega}{2})^{\frac{1}{2}}\,(\sin\omega)^{k-\frac{1}{2}}\,d\omega\right]$$

Now

$$(n+1)^{\frac{1}{2}}\int_{\alpha_n}^{\pi-n^{-1}} e^{i(n+1)\,\omega}\,F(\omega)\,(\cot\tfrac{\omega}{2})^{\frac{1}{2}}\,(\sin\omega)^{k-\frac{1}{2}}\,d\omega$$

$$= \tfrac{1}{2}\,(n+1)^{\frac{1}{2}}\Bigg[\int_{\alpha_n}^{\pi-n^{-1}} e^{i(n+1)\omega}\,F(\omega)\,(\cot\tfrac{\omega}{2})^{\frac{1}{2}}\,(\sin\omega)^{k-\frac{1}{2}}\,d\omega$$

$$= \int_{\alpha_n-\mu_n}^{\pi-\mu_n-n^{-1}} e^{i(n+1)\omega}\,F(\omega+\mu_n)\left(\cot\frac{\omega+\mu_n}{2}\right)^{\frac{1}{2}}(\sin(\omega+\mu_n))^{k-\frac{1}{2}}\,d\omega\ \Bigg],$$

$$(\mu_n = \pi/n+1)$$

and this is consequently less in modulus than

$$\tfrac{1}{2}\,(J_1+J_2+J_3+J_4),$$

where

$$J_1 = (n+1)^{\frac{1}{2}}\int_{\alpha_n-\mu_n}^{\alpha_n}\left|\,F(\omega+\mu_n)\left(\cot\left(\frac{\omega+\mu_n}{2}\right)^{\frac{1}{2}}\left(\sin(\omega+\mu_n)\right)^{k-\frac{1}{2}}\right|\,d\omega,$$

$$J_2 = (n+1)^{\frac{1}{2}}\int_{\pi-n^{-1}-\mu_n}^{\pi-n^{-1}}\left|F(\omega)\,(\cot\tfrac{\omega}{2})^{\frac{1}{2}}\,(\sin\omega)^{k-\frac{1}{2}}\right|d\omega,$$

$$J_3 = (n+1)^{\frac{1}{2}}\int_{\alpha_n}^{\pi-n^{-1}-\mu_n}\left|F(\omega+\mu_n)-F(\omega)\right|\left(\cot\left(\frac{\omega+\mu_n}{2}\right)\right)^{\frac{1}{2}}(\text{in}\,(\omega+\mu_n))^{k-\frac{1}{2}}\,d\omega,$$

$$J_4 = (n+1)^{\frac{1}{2}}\int_{\alpha_n}^{\pi-n^{-1}-\mu_n}\left|\left(\cot\left(\frac{\omega+\mu_n}{2}\right)\right)^{\frac{1}{2}}(\sin(\omega+\mu_n))^{k-\frac{1}{2}}-\left(\cot\frac{\omega}{2}\right)^{\frac{1}{2}}(\sin\omega)^{k-\frac{1}{2}}\right|\,F(\omega)\,d\omega,$$

Now,

$$(4{\cdot}7{\cdot}5_1)\qquad J_1 = O(n^{\frac{1}{2}})\ \alpha_n^{\frac{1}{2}-k}.\ \alpha_n^{k-1}.\ \mu_n,$$

$$= O(1).$$

$$(4{\cdot}7{\cdot}5_2)\qquad J_2 = O(n^{\frac{1}{2}})\int_{\pi-n^{-1}-\mu_n}^{\pi-n^{-1}}\omega^{\frac{1}{2}-k}\,\omega^{k-1}\,d\omega,$$

$$= O(1)$$

$$(4{\cdot}7{\cdot}5_3)\qquad J_3 = O(n^{\frac{1}{2}})\int_{\alpha_n}^{\pi}[0(\mu_n)^{\frac{1}{2}-k}.\ \omega^{k-1}]\,d\omega,$$

$$= O(n^{k})$$

$$(4{\cdot}7{\cdot}5_4)\qquad J_4 = O(n^{\frac{1}{2}})\int_{\alpha_n}^{\pi}\mu_n\,\omega^{k-2}.\ \omega^{\frac{1}{2}-k}\,d\omega,$$

$$= O(1).$$

Summing up we see that the theorem is proved for the case $0 < k < \frac{1}{2}$.

To prove the theorem for the case $k = 0$, it is sufficient to show that $J_3 = 0$ ($\log n$), since the value of k does not effect the other results.

Now when $k = 0$, we have from ($4{\cdot}7{\cdot}5_3$)

$$J_3 = 0(n^{\frac{1}{2}}) \int_{\alpha_n}^{\pi} \mu_n^{\frac{1}{2}} \omega^{-1} \, d\omega \tag{4·8}$$

$$= 0(\log n)$$

Thus the theorem is completely proved

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References

1. Beohar, B. K. On order of partial sum of Legendre series. *Chap. VIII of the thesis presented for the decree of Doctor of Philosophy, Vikram Univ., Ujjain (India)* 1966.
2. Du Plessis, N. The Cesàro summability of Laplace series. *Journ. Lond. Math. Soc.*, **27**, 337-352, 1952.
3. Kogbetliantz, E. Recherches Sur la sommabilité des séries ultrasphériques des moyennes arithmétiques, *Journ. de Math.*, **3**(9), 107-187, 1924.
4. Sansone, G. Orthogonal Functions. (*English Edition*), *New York*, 1959.
5. Szàsz, O. On strong summability of Fourier series. *Trans. Amer. Math. Soc.*, **48**, 117-125, 1940.
6. Wilson, B. M. On the manner of divergence of Legendre series of an integrable functions. *Proc. Lond. Math. Soc.*, **2**, 389-400, 1923.

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Effect of Viscosity on the Stability of Superposed Fluids

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Abstract

A study of the effect of viscosity on the stability of the superposed fluids is made and a dispersion relation is derived in case of two uniform fluids of constant densities and velocities. Kelvin-Helmoltz and Rayleigh-Taylor instabilities are discussed and it is found that these instabilities are independent of the viscosity.

Introduction

S. Chandrasekher[1] and recently Sharma and Srivastava[2] have discussed the Kelvin-Helmoltz instability in the presence of a magnetic field for non viscous fluids. We derive here a dispersion relation for the superposed viscous fluids and discuss Kelvin-Helmoltz and Rayleigh-Taylor instabilities. It is first supposed that the fluids are in relative motion to obtain dispersion relation in case of Kelvin-Helmoltz instability and the fluids are set at rest to discuss Rayleigh-Taylor instability.

The fluids are supposed to be viscous incompressible and moving with velocity U in the x-direction and separated by a horizontal boundary $z = 0$. Let $\delta\rho$ and δp be a small disturbance in density and pressure respectively such that the density at any point (x, y, z) be $\rho + \delta\rho$ and pressure $p + \delta p$. The components of velocity in the perturbed state be $U + u, v, w$. The perturbed equation of fluids are :

$$\rho\frac{\partial u}{\partial t} + \rho U\frac{\partial u}{\partial x} + \rho w\frac{dU}{dz} = -\frac{\partial}{\partial x}(\delta p) + \mu\nabla u \tag{1}$$

$$\rho\frac{\partial v}{\partial t} + \rho U\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}(\delta p) + \mu\nabla^2 v \tag{2}$$

$$\rho\frac{\partial w}{\partial t} + \rho U\frac{\partial w}{\partial x} = -\frac{\partial}{\partial z}(\delta p) - g\delta\rho + \Sigma_s\ T_s\left[\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\delta z_s\right]\delta(z - z_s) + \mu\nabla^2 w \tag{3}$$

where the coefficient of viscosity μ is supposed to be constant and the other symbols have their usual meaning.

Equation of continuity, for this case is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{4}$$

Also the density of every particle remains unchanged and δz can be expressed in terms of the normal component of the velocity w_s at z_s, corresponding perturbation equations are

$$\frac{\partial(\delta\rho)}{\partial t} + U\frac{\partial}{\partial x}(\delta\rho) = -w\frac{d\rho}{dz} \tag{5}$$

$$\frac{\partial(\delta z_s)}{\partial t} + U_s\frac{\partial}{\partial x}(\delta z_s) = w(z_s) \tag{6}$$

Derivation of dispersion relations

Following S. Chandrasekhar we analyse the disturbance into normal modes dependent on x, y, and t, which is given by $\exp i(k_x x + k_y y + nt)$. For solutions having this dependence on x, y, and t, equations (1) to (6) become,

$$i\rho n'u + \rho(DU)w = -ik_x\,\delta p + \mu(D^2 - k^2)u \tag{7}$$

$$i\rho n'v = -ik_y\,\delta p + \mu(D^2 - k^2)v \tag{8}$$

$$i\rho n'w = -D(\delta p) - g\delta\rho - k^2 \Sigma\, T_s z_s\,\delta(z - z_s) + \mu(D^2 - k^2)w \tag{9}$$

$$ik_x u + ik_y v + Dw = 0 \tag{10}$$

$$in'\delta\rho = -wD\rho \tag{11}$$

$$in'\delta z_s = w(z_s) \tag{12}$$

where $D = \frac{d}{dz}$; $n = n + k_x U$.

Multiplying equation (7) by $-ik_x$ and equation (8) by $-ik_y$ and adding and then eliminating δp between this equation and equation (9), with the help of equations (10) (11) and (12), we obtain

$$\begin{aligned} &D[\,\rho n'Dw - \rho k_x(DU)w\,] + D[\,i\mu(D^2 - k^2)\,Dw\,] - k^2\rho n'w \\ &= gk^2(D\rho)\frac{w}{n'} + ik^2\mu(D^2 - k^2)\,w - k^4\left[T_s\left(\frac{w}{n'}\right)_s\delta(z - z_s)\right] \end{aligned} \tag{13}$$

Here w/n' is continuous, hence integrating equation (13) between $z - \varepsilon$ and $z + \varepsilon$ and in the limit $\varepsilon \to 0$, we get

$$\Delta_s[\,\rho n'Dw - \rho k_x(DU)\,w + i\mu\,(D^2 - k^2)\,Dw\,] = gk^2\left[\Delta_s\rho - \frac{k^2}{g}T_s\right]\left(\frac{\omega}{n'}\right)_s \tag{14}$$

where Δ_s has its usual meaning.

In case of constant density ρ and constant streaming U, equation (13) becomes

$$D[\,\rho n'Dw + i\mu(D^2 - k^2)\,Dw\,] - k^2\rho n'w = ik^2\mu(D^2 - k^2)\,w$$

which can be written as

$$(D^2 - k^2)\,(D^2 - q^2)\,w = 0 \tag{15}$$

where $q^2 = k^2 + \frac{in'}{\nu}$; $\nu = \frac{\mu}{\rho}$ (kinematic viscosity).

The two fluids of densities ρ_1 and ρ_2 are separated by a horizontal boundary at $z = 0$, and the density of the upper fluids is ρ_2 and since w/n' is continuous, w cannot grow exponentially and further if the real part of q is positive, the solutions for the two regions of the fluid, can be written as

$$w_1 = (Ae^{kz} + Be^{q_1 z})\,n_1' \qquad \text{for } z < 0$$

$$w_2 = (Ae^{-kz} + Be^{-q_2 z})\, n_2' \quad \text{for } z > 0$$

where $n_j' = n + k_x U_j$; $q_j^2 = k^2 + \frac{in_j'}{\nu_j}$ for $j = 1, 2$

$\mu(D^2 + k^2)\, \frac{w}{n'}$, is also continuous[3], hence we can have

$$\mu_1[2k^2A + (q_1^2 + k^2)\, B] = \mu_2[2k^2A + (q_2^2 + k^2)\, B]$$

We obtain $B = N.\, A$, where $N = \frac{2k^2(\mu_1 - \mu_2)}{2k^2(\mu_2 - \mu_1) + i\,(n_2'\,\rho_2 - n_1',\, \rho_1)}$

Thus the solutions of w are given by

$$w_1 = An_1'\,(e^{kz} + Ne^{q_1 z}) \qquad \text{for } z < 0$$

$$w_2 = An_2'\,(e^{-kz} + Ne^{-q_2 z}) \text{ for } z > 0$$

Substituting w_1 and w_2 in equation (14), we obtain

$$\rho_2\, n_2'^2 + \rho_1\, n_1'^2 = gk^2\left[\rho_1 - \rho_2 + \frac{k^2}{g} T\right][1 + N]$$

Putting the value of N from above we obtain, the dispersion relation

(16)
$$\{ 2k^2(\mu_2 - \mu_1) + i(\rho_2 n_2' - \rho_1 n_1') \}\, (\rho_2 n_2'^2 + \rho_1 n_1'^2)$$
$$= igk^2 \left[\rho_1 - \rho_2 + \frac{k^2}{g}\, T\right][\, \rho_2 n_2' - \rho_1 n_1'\,]$$

Discussion

We now discuss Kelvin-Helmolts and Rayleigh-Taylor instabilities.

(*i*) *Kelvin-Helmoltz instability* : In this we consider the cases of two liquids of same viscosity and of different viscosity.

(*a*) *Two liquids of same viscosity* : Here we have $\mu_1 = \mu_2$, putting this condition in equation (16), we obtain

$$\rho_2 n_2'^2 + \rho_1 n_1'^2 = gk^2\left[\rho_1 - \rho_2 + \frac{k^2}{g} T\right]$$

For n real we obtain the following stability condition

(17)
$$k_x^2(U_1 - U_2)^2 < \frac{gk}{\alpha_1 \alpha_2}\left\{ (\alpha_1 - \alpha_2) + \frac{k^2\, T}{g(\rho_1 + \rho_2)} \right\}$$

where
$$\alpha_1 = \frac{\rho_1}{\rho_1 + \rho_2} \text{ and } \alpha_2 = \frac{\rho_2}{\rho_1 + \rho_2}$$

(*b*) *Two fluids of different viscosity* : Here we have $\mu_1 \neq \mu_2$, separating equation (16) into real and imaginary parts, we obtain

(18)
$$2k^2(\mu_2 - \mu_1)\, (\rho_2 n_2'^2 + \rho_1 n_1'^2) = 0$$

or
$$\rho_2 n_2'^2 + \rho_1 n_1'^2 = 0$$

(19)
$$\rho_2 n_2'^2 + \rho_1 n_1'^2 = gk^2\left[\rho_1 - \rho_2 + \frac{k^2}{g}\, T\right]$$

For n to be real equation (18) gives $U_1 = U_2$ and now putting this in equation (19) we obtain

either $$\rho_1 > \rho_2 \quad \text{or} \quad \alpha_1 + \frac{k^2 T}{g(\rho_1 + \rho_2)} > \alpha^2$$

(20) *i.e* $$\rho_1 > \rho_2 \text{ or } k < k_c$$

where $k_c = \left\{ (\rho_2 - \rho_1) \frac{g}{T} \right\}^{1/2}$ and k_c is called the critical value of k.

If we solve equation (19) for real value of n, we obtain the same condition as given in equation (17), putting this condition in (18) we obtain the stability condition for $U_1 \neq U_2$

(21) $$\rho_2 > \rho_1 + \frac{k^2 T}{g} \quad \text{or} \quad k < k_c$$

(*ii*) *Rayleigh-Taylor instability* : We put $U = 0$ and then $n' = n$ in equation (16) and since w/n and $\mu(D^2 + k^2)\frac{w}{n}$ are continuous, we obtain the dispersion relation for the case as

(22) $$\{ 2k^2(\mu_2 - \mu_1) + i(\rho_2 - \rho_1)\, n \} (\rho_2 + \rho_1) n = igk^2 \left[\rho_1 - \rho_2 + \frac{k^2}{g} \right] (\rho_2 - \rho_1)$$

(*a*) *Two liquids of same viscosity* : Here we have $\mu_1 = \mu_2$, equation (22) becomes

$$n^2 = gk^2 \left[\alpha_1 - \alpha_2 + \frac{k^2 T}{(\rho_1 + \rho_2) g} \right]$$

For stability, n must be real, which leads to following conditions

(*i*) either $\rho_1 > \rho_2$ or (*ii*) $k < k_c$; where $k_c = \left\{ (\rho_2 - \rho_1) \frac{g}{T} \right\}^{1/2}$

which are the same as given by Chandrasekher[4].

(*b*) *Two liquids of different viscosity* : Here we have $\mu_1 \neq \mu_2$. Equating real and imaginary parts of the equation (22), we obtain stability condition as given in case (*ii*), (*a*).

We therefore conclude that the stability of superposed fluids is independent of viscosity.

References

1. Chandrasekher, S. *Oxford, Clarendon Press, Ch. XII,* 1961.
2. Sharma, R. C. and Srivastava, K. M. *No. 9, Canadian Journal of Physics,* 47, pp. 975, 1969.
3. Chandrasekher, S. *Pro. Camb. Phil. Soc.*, 51, 162–78, 1955.
4. Chandrasekher, S. *Oxford, Clarendon Press,* pp. 435, 1961.

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Series Involving Legendre Polynomials

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1. Rainville [7] expressed Legendre polynomials of one variable in the form of Legendre polynomials of another variable as follows,

$$(1\cdot1)\qquad P_n(\cos\alpha) = \left(\frac{\sin\alpha}{\sin\beta}\right)^n \sum_{k=o}^{n} \binom{n}{k} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^{n-k} P_k(\cos\beta).$$

Bhonsle [1] made the substitution $\beta = 2\alpha$ in the above relation and used it for evaluating a large number of interesting integrals.

An attempt has been made to obtain a similar series which has been used later on for evaluating certain integrals which are believed to be new. The following relation of Erdélyi [5, p. 264], has been considered for obtaining the series,

$$(1\cdot2)\qquad (1-2\,t\,x+t^2)^{-\frac{1+\nu}{2}} P_\nu^\mu \left[\frac{x-t}{(1-2xt+t^2)^{\frac12}}\right] = \sum_{r=o}^{\infty} \binom{\nu-\mu+r}{r} P^{\mu}_{\nu+r}(x)t^r.$$

At the end, in the article 4, an application to a heat conduction problem is considered.

2. In (1·2), substituting

$$z = \cos\beta,\ t = \frac{\cos(\beta-\alpha)}{\cos\alpha}$$

and therefore,

$$x - t = -\frac{\sin\beta\sin\alpha}{\cos\alpha} \text{ and } (1-2\,t\,x+t^2) = \frac{\sin^2\beta}{\cos^2\alpha},$$

we obtain,

$$(2\cdot1)\qquad \left[\frac{\sin\beta}{\cos\alpha}\right]^{-(1+\nu)} P_\nu^\mu(-\sin\alpha) = \sum_{r=o}^{\infty} \binom{\nu-\mu+r}{r} P^{\mu}_{\nu+r}(\cos\beta)\left[\frac{\cos(\beta-\alpha)}{\cos\alpha}\right]^r.$$

Let $\mu = m$, $\nu = m+n$, wet get,

$$(2\cdot2)\qquad P^m_{m+n}(-\sin\alpha) = \left[\frac{\sin\beta}{\cos\alpha}\right]^{m+n+1} \sum_{r=o}^{\infty} \binom{n+r}{r} \left[\frac{\cos(\beta-\alpha)}{\cos\alpha}\right]^r P^m_{m+n+r}(\cos\beta).$$

Let $m = 0$ in (2·2), so that

$$(2\cdot3)\qquad P_n(-\sin\alpha) = \left(\frac{\sin\beta}{\cos\alpha}\right)^{n+1} \sum_{r=0}^{\infty} \binom{n+r}{r} \left[\frac{\cos(\beta-\alpha)}{\cos\alpha}\right]^{r} P_{n+r}(\cos\beta).$$

But $P_n(-\sin\alpha) = (-1)^n P_n(\sin\alpha)$, therefore (2·3) becomes

$$(2\cdot4)\qquad P_n(\sin\alpha) = (-1)^n \left(\frac{\sin\beta}{\cos\alpha}\right)^{n+1} \sum_{r=0}^{\infty} \binom{n+r}{r} \left[\frac{\cos(\beta-\alpha)}{\cos\alpha}\right]^{r} P_{n+r}(\cos\beta).$$

Let $\beta = 2\alpha$ in (2·4) and we get,

$$(2\cdot5)\qquad P_n(\sin\alpha) = (-1)^n (2\sin\alpha)^{n+1} \sum_{r=0}^{\infty} \binom{n+r}{r} P_{n+r}(\cos 2\alpha).$$

Writing $\cos 2\alpha = x$ and $\sin\alpha = \sqrt{\frac{1-x}{2}}$, we have

$$(2\cdot6)\qquad \frac{(-1)^n}{2^{n+1}\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}} P_n\left(\sqrt{\frac{1-x}{2}}\right) = \sum_{r=0}^{\infty} \binom{n+r}{r} P_{n+r}(x).$$

3. The following integrals have been evaluated, by making use of (2·6).

Using the orthogonal property [4, p. 170 ; (8), (10)] for Legendre polynomials, we have

$$(3\cdot1)\qquad \int_{-1}^{1} 2^{-\frac{n+1}{2}} (1-x)^{-\frac{n+1}{2}} P_n\left(\sqrt{\frac{1-x}{2}}\right) P_r(x)\,dx \begin{cases} = 0, & r < n \\ = (-1)^n \binom{r}{r-n} \dfrac{2}{(2r+1)}, & r \geqslant n \end{cases}$$

If p and q are + *ve* integers and $q \leqslant p$, we have the Adam's expansion [10, p. 331].

$$(3\cdot2)\qquad P_p(\mu)\,P_q(\mu) = \sum_{r=0}^{q} \frac{A_{q-r}\,A_r\,A_{p-r}}{A_{p+q-r}} \left(\frac{2p+2q-4r+1}{2p+2q-2r+1}\right) P_{p+q-2r}(\mu),$$

where $A_r = \dfrac{1.\,3.\,5.\ldots\ldots(2r-1)}{r\,!}$.

Thus making use of the result (3·2) with (2·6), we obtain, when $p+q-2r \geqslant n$,

$$(3\cdot3)\qquad \int_{-1}^{1} \frac{(-1)^n P_n\left(\sqrt{\frac{1-x}{2}}\right)}{\left(\sqrt{\frac{1-x}{2}}\right)^{n+1}} P_p(x)\,P_q(x)\,dx = \sum_{r=0}^{q} \frac{A_{q-r}\,A_r\,A_{p-r}\,.\,2^{n+2}}{A_{p+q-r}\,(2p+2q-2r+1)}$$

$$\binom{p+q-2r}{p+q-2r-n} = \frac{2^{n+1}\,(\frac{1}{2})_p\,(\frac{1}{2})_q\,(p+q)\,!\,(p+q)\,!}{(\frac{1}{2})_{p+q+1}\,p\,!\,q\,!\,n\,!\,(p+q-n)\,!}$$

$$\times {}_8F_5\left[\begin{matrix} -p, -q, -(p+q+\frac{1}{2}), -\frac{(p+q-n)}{2}, -\frac{1}{2}(p+q-n-1), \frac{1}{2}; \\ \frac{1}{2}-p, \frac{1}{2}-q, -(p+q), -\frac{1}{2}(p+q), -\frac{1}{2}(p+q-1); \end{matrix} \; 1\right].$$

Following the method of Shabde [8], if

$$p_1 + p_2 + p_3 + \dots\dots + p_r = s,$$

and $\sqrt{\frac{1-x}{2}} = y$, we have,

$$(3{\cdot}5) \qquad \int_{-1}^{1} \frac{(-1)^s}{(2y)^{s+r-1}} \cdot P_{p_1}(y)\, P_{p_2}(y) \dots\dots P_{p_r}(y) .\, P_s(1-2y^2)\, dy$$

$$= \frac{(s\,!)^2}{(2s+1)!} \prod_{t=1}^{r} \frac{(2p_t)\,!}{(p_t)\,!\,(p_t)\,!}$$

Grosswald's formula [1] for the r th differentiation of $P_n(x)$ when $x = 1$ is given by

$$(3{\cdot}6) \qquad \left[\left(\frac{d}{dx}\right)^r P_n(x) \right]_{x=1} = \frac{(n+r)\,!}{2^r\, r\,!\,(n-r)\,!}.$$

Thus differentiating (2·6) r times with respect to x and using the above formula, we have

$$(3{\cdot}7) \qquad \left(\frac{d}{dx}\right)^r \left[\frac{(-1)^n}{\left(\sqrt{\frac{1-x}{2}}\right)^{n+1}} P_n\left(\sqrt{\frac{1-x}{2}}\right) \right]_{x=1}$$

$$= 2^{n+1-r} \binom{n+r}{r} \frac{(n+2r)\,!}{n\,!\,r\,!} \; {}_3F_2\left[\begin{matrix} 1, n+r+1, n+2r+1; \\ r+1, n+1; \end{matrix} \; 1\right]$$

By using Neumann's formula [10, p. 320]

$$Q_n(\mu) = \frac{1}{2} \int_{-1}^{1} \frac{P_n(y)}{\mu - y}\, dy$$

with (2·6), we get

$$(3{\cdot}8) \qquad \int_{-1}^{1} \frac{(-1)^n}{\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}} \cdot \frac{P_n\left(\sqrt{\frac{1-x}{2}}\right)}{y - x}\, dx = 2^{n+2} \sum_{k=0}^{\infty} \binom{n+k}{k} Q_{n+k}(x].$$

From Erdelyi [4, p. 171], when $\sigma > -1$, we have

$$(3{\cdot}9) \qquad \int_0^1 x^{\sigma} P_n(x)\, dx = \frac{\sqrt{\pi}\, 2^{-\sigma-1}\, \Gamma(1+\sigma)}{\Gamma\left(1 + \frac{\sigma}{2} - \frac{n}{2}\right) \Gamma\left(\frac{3}{2} + \frac{\sigma}{2} + \frac{n}{2}\right)}.$$

Using the above result (3·9) with (2·6), we get

$$(3\cdot 10)\qquad \int_0^1 \frac{x^{\sigma}(-1)^n}{2^{n+1}\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}} P_n\left(\sqrt{\frac{1-x}{2}}\right) {}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots\ldots, \alpha_p\,; \\ \beta_1, \beta_2, \ldots\ldots, \beta_q\,;\end{matrix}\ \pm x^2 t^2\right] dx$$

$$= \sum_{k=0}^{\infty} \binom{n+k}{k} \frac{\Gamma\left(\frac{1+\sigma}{2}\right)\Gamma\left(\frac{2+\sigma}{2}\right)}{2\Gamma\left(1+\frac{\sigma}{2}-\frac{n+k}{2}\right)\Gamma\left(\frac{3+\sigma+n+k}{2}\right)}$$

$${}_{p+2}F_{q+2}\left[\begin{matrix}\alpha_1, \alpha_2, \ldots\ldots, \alpha_p, 1+\frac{\sigma}{2}, \frac{1}{2}+\frac{\sigma}{2}\,; \\ \beta_1, \beta_2, \ldots\ldots, \beta_q, 1+\frac{\sigma}{2}-\frac{n+k}{2}, \frac{3+\sigma+n+k}{2}\end{matrix}\ \pm t^2\right].$$

$$|t| < 1 \text{ and } p = q+1$$

The hypergeometric series should be terminating when $p > q+1$. We can obtain many particular cases of the results (3·10).

We have due to Brafman [7, p. 168], the following generating function for Legendra polynomials,

$$(3\cdot 11)\qquad {}_2F_1\left[\begin{matrix}a, 1-a\,; \\ 1\,;\end{matrix}\ \frac{1-t-\rho}{2}\right] \times {}_2F_1\left[\begin{matrix}a, 1-a\,; \\ 1\,;\end{matrix}\ \frac{1+t-\rho}{2}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (1-a)_n t^n}{n!\,n!} P_n(x),$$

where $\rho = (1-2xt+t^2)^{\frac{1}{2}}$.

Using the result (3·11) with (2·6), we get

$$(3\cdot 12)\qquad \int_{-1}^1 \frac{(-1)^n}{2^{n+1}\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}} P_n\left(\sqrt{\frac{1-x}{2}}\right) {}_2F_1\left[\begin{matrix}a, 1-a\,; \\ 1\,;\end{matrix}\ \frac{1-t-\rho}{2}\right] \times {}_2F_1\left[\begin{matrix}a, 1-a\,; \\ 1\,;\end{matrix}\ \frac{1+t\ \rho}{2}\right] dx$$

$$= \frac{(a)_n (1-a)_n \cdot 2 \cdot t^n}{n!\,n!\,(2n+1)}\ {}_3F_2\left[\begin{matrix}a+n, 1-a+n, \frac{1}{2}+n\,; \\ 1+n, \quad \frac{3}{2}+n\,;\end{matrix}\ t\right].$$

If $a = -m$, we have

$$(3\cdot 13)\qquad \int_{-1}^1 \frac{(-1)^n}{2^{n+1}\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}} P_n\left(\sqrt{\frac{1-x}{2}}\right) P_m(t-\rho)\, P_m(t+\rho)\, dx$$

$$= \frac{(-m)_n\,(1+m)_n\,.\,2.\,t^n}{n\,!\,n\,!\,(2n+1)}\,.\,{}_3F_2\left[\begin{matrix}-m+n,\,1+m+n,\,n+\frac{1}{2}\,;\\ n+1,\,n+\frac{3}{2}\,;\end{matrix}\;t\right]$$

We have from [7, p. 164(2)]

$$(3{\cdot}14)\qquad (1-xt)^{-b}\,{}_2F_1\left[\begin{matrix}\frac{b}{2}\,,\;\frac{b+1}{2}\;;\\ 1\;;\end{matrix}\;\frac{t^2(x^2-1)}{(1-xt)^2}\right] = \sum_{n=0}^{\infty}\frac{(b)_n\,P_n(x)}{n\,!}\,t^n.$$

Using (3·14) with (2·6), we get

$$(3{\cdot}15)\qquad \int_{-1}^{1}\frac{(-1)^n\,(1-xt)^{-b}}{2^{n+1}\left(\frac{1-x}{2}\right)^{\frac{n+1}{2}}}\;P_n\!\left(\sqrt{\frac{1-x}{2}}\right){}_2F_1\left[\begin{matrix}\frac{b}{2}\,,\;\frac{b+1}{2}\;;\\ 1\;;\end{matrix}\;\frac{t^2(x^2-1)}{(1-xt)^2}\right]dx$$

$$= \frac{2(b)_n\,t^n}{n\,!\,(2n+1)}\,.\,{}_2F_1\left[\begin{matrix}b+n,\,n+\frac{1}{2}\,;\\ n+\frac{3}{2}\,;\end{matrix}\;t\right]$$

We have also due to Mitra [1],

$$(3{\cdot}16)\qquad \int_0^1 P_n(1-2y^2)\,J_0(yx)\,y\,dy = \frac{1}{x}J_{2n+1}(x).$$

Substituting y for $\sqrt{\frac{1-x}{2}}$ in (2·6), we get

$$\frac{(-1)^n}{2^{n+1}\,y^{n+1}}\,P_n(y) = \sum_{k=0}^{\infty}\binom{n+k}{k}P_{n+k}\,(1-2y^2).$$

Combining the above relation with (3·16), we get

$$(3{\cdot}17)\qquad \int_0^1 \frac{(-1)^n}{2^{n+1}\,y^{n+1}}\,.\,P_n(y)\,J_0(yx)\,y\,dy$$

$$= \sum_{k=0}^{\infty}\binom{n+k}{k}\frac{1}{x}J_{2n+2k+1}\,(x)$$

Expanding $J_o\,(yx)$ in terms of yx and integrating the left hand side of (3·17) term by term, with the help of (3·9), we get

$$(3{\cdot}18)\qquad \int_0^1 \frac{(-1)^n}{2^{n+1}\,y^n}\,P_n\,(y)\,J_o\,(y\,x)\;dy$$

$$= \frac{(-1)^n}{2}\sum_{k=o}^{\infty}\frac{\left(\frac{1}{2}-\frac{n}{2}\right)_k\left(1-\frac{n}{2}\right)_k}{k\,!\,(1)_k\,(1-n)_k\,(\frac{3}{2})_k}\left(-\frac{x^2}{4}\right)^k$$

$$= \frac{(-1)^n}{2}\; {}_2F_3\left[\begin{matrix} \frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} & : & - \frac{x^2}{k} \\ 1, 1 - n, \frac{3}{2} & ; & \end{matrix}\right]$$

Thus equating (3·17) and (3·18), we get

$$(3{\cdot}19) \qquad \frac{2}{x} \sum_{k=0}^{\infty} \binom{n+k}{k} J_{2n+2k+1}(x) = (-1)^n\; {}_2F_3\left[\begin{matrix} \frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} ; & - \frac{x^2}{4} \\ 1, 1 - n, \frac{3}{2} \;\; ; & \end{matrix}\right]$$

Using the following orthogonal property of Bessel functions, for $\nu > -1$,

$$(3{\cdot}20) \qquad \int_0^{\infty} t^{-1} J_{\nu+2n+1}(t) \cdot J_{\nu+2m+1}(t)\, dt \begin{cases} = 0, & m \neq n \\ = \dfrac{1}{2(2n+\nu+1)}, & m = n \end{cases}$$

with (3·19), we get,

$$(3{\cdot}21) \qquad \int_0^{\infty} (-1)^n\; {}_2F_3\left[\begin{matrix} \frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} ; & - \frac{x^2}{4} \\ 1, 1 - n, \frac{3}{2} \;\; ; & \end{matrix}\right] J_{2n+2k+1}(x)\, dx$$

$$= \binom{n+k}{k} \frac{1}{(2n+2k+1)}.$$

Also we have,

$$(3{\cdot}22) \qquad \int_0^{\infty} x\,(-1)^{2n} \left\{ {}_2F_3\left[\begin{matrix} \frac{1}{2} - \frac{n}{2}, 1 - \frac{n}{2} ; & - \frac{x^2}{4} \\ 1, 1 - n, \frac{3}{2} \;; & \end{matrix}\right] \right\}^2 dx$$

$$= \int_0^{\infty} \frac{4}{x} \left[\sum_{k=0}^{\infty} \binom{n+k}{k} J_{2n+2k+1}(x) \right]^2 dx$$

$$= 2 \sum_{k=0}^{\infty} \frac{(n+k)!\,(n+k)!}{k!\,k!\,n!\,n!}, \frac{1}{(2n+2k+1)}$$

$$= \frac{2}{(2n+1)}\; {}_3F_2\left[\begin{matrix} n+1, n+1, n+\frac{1}{2} ; & 1 \\ 1, n+\frac{3}{2} : & \end{matrix}\right]$$

4. *An application to a Heat Conduction problem.*

Churchill [2] considered the problem given as follows. If $U(x, t)$ represents temperature in a non-homogeneous insulated bar, along the axis of x, with its ends at $x = -1$ and $x = +1$, in which the thermal conductivity is proportional to $(1 - x^2)$, the heat equation takes the form,

$$(4{\cdot}1) \qquad \frac{\partial U}{\partial t} = b \frac{\partial}{\partial x}\left[(1 - x^2) \frac{\partial U}{\partial x}\right]$$

where $b > 0$ and constant, if the thermal coefficient C_0 is constant. The ends $x = \pm 1$ are insulated because the conductivity vanishes there.

If $\quad U(x, 0) = f(x), \qquad -1 < x < 1$

the solution of the above equation (4·1) is given by

$$(4\cdot2) \qquad U(x, t) = \sum_{n=o}^{\infty} \frac{(2n+1)}{2}\, exp\,[-n\,(n+1)\,bt\,]\, P_n\,(x) \int_{-1}^{1} f\,(\xi)\, P_n\,(\xi)\, d\,\xi.$$

The equation (4·1) can also be solved with the application of Legendre transform. This transform was first introduced by Tranter [9] in 1950 and was later on used by Churchill [3], and Normington and Blackwell [6] in solving the problem on heat transfer.

Let there be a function $f(x)$ which can be expanded in Legendre polynomials as follows,

$$(4\cdot3) \qquad f(x) = \sum_{n=o}^{\infty} a_n\, P_n\,(x).$$

Legendre transforms are defined as

$$(4\cdot4) \qquad \bar{f}\,(n) = \int_{-1}^{1} f\,(x)\, P_n\,(x)\, dx$$

with the inversion theorem being given by

$$(4\cdot5) \qquad f\,(x) = \tfrac{1}{2} \sum_{n=o}^{\infty} (2n+1)\, \bar{f}\,(n)\;\; P_n\,(x).$$

Applying Legendre transform to (4·1) and making use of Legendre's equation,

$$(4\cdot6) \qquad \frac{d}{dx}\left\{ (1-x^2)\,\frac{d}{dx}\, P_n\,(x) \right\} = -\,n\,(n+1)\, P_n\,(x),$$

we obtain finally,

$$(4\cdot7) \qquad \frac{d}{dt}\,\overline{U}\,(n) = -\,b\;n\;\;(n+1)\;\;\overline{U}\,(n)$$

The solution of (4·7) is given by

$$(4\cdot8) \qquad \overline{U}\,(n) = A\; e^{-bn(n+1)t}$$

The constant A can be evaluated from the initial condition,

$$U\,(x, o) = f\,(x)$$

Hence, $[\,\overline{U}\,(n)]_{t=o} = \bar{f}\,(n)$.

Therefore (4·8) becomes

$$(4\cdot9) \qquad \overline{U}\,(n) = \bar{f}\;(n)\; e^{-bn(n+1)t}$$

The inversion theorem (4·5) will therefore give

$$(4\cdot10) \qquad U\,(x, t) = \tfrac{1}{2} \sum_{n=o}^{\infty} (2n+1)\; \bar{f}\,(n)\; e^{-bn(n+1)t}\;\; P_n\,(x).$$

A function $f(x)$, which will suitably serve as initial temperature distributions is (2·6), expressed as

$$f_m(x) = (-1)^m \, 2^{-\frac{m+1}{2}} \, (1-x)^{-\frac{m+1}{2}} \, P_m\left(\sqrt{\frac{1-x}{2}}\right). \tag{4·11}$$

From (3·1) we know that

$$\int_{-1}^{1} (-1)^m \, 2^{-\frac{m+1}{2}} \, (1-x)^{-\frac{m+1}{2}} \, P_m\left(\sqrt{\frac{1-x}{2}}\right) P_n(x)\, dx \begin{cases} = 0, \; n < m \\ = \binom{n}{n-m} \dfrac{2}{(2n+1)}, \; n \geqslant m. \end{cases}$$

Therefore, we have

$$\bar{f}_m(n) = \binom{n}{n-m} \frac{2}{(2n+1)}, \text{ for } n \geqslant m. \tag{4·12}$$

Thus from (4·10), the solution for the function $f(x)$ given by (4·11), will be,

$$U_m(x_1 t) = \sum_{n=0}^{\infty} \binom{n}{n-m} e^{-bn(n+1)t} P_n(x). \tag{4·13}$$

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References

1. Bhonsle, B. R. On a series of Rainville involving Legendre polynomials. *Proc. Amer. Math. Soc.*, **8**(1), 10–14, 1957.
2. Churchill, R. V. Fourier Series and Boundary values problems. *McGraw Hill Book Co.*, second edition, p. 224, 1963.
3. Churchill, R. V. The operational calculus of Legendre Transforms. *Jour. of Maths. and Physics*, **33**, 165–177, 1954.
4. Erdélyi, A. Editor. Higher Transcendental Functions, Vol. I p. 170–171,
5. Erdélyi, A. Editor. Higher Transcendental Functions, Vol. III, p. 264.
6. Normington, E. J. and Blackwell, J. H. Transient heat flow from constant temperature Spheroids and the thin circular disk. *The Quart. Jour. Mech. and Appl. Maths.*, **17**(1), 1964.
7. Rainville. E. D. Special Functions. *The MacMillan Company, New York*, 168, 1963.
8. Shabde, N. G. On some results involving Legendre Functions. *Proc. Benares Math. Soc.*, **7**(1), New series, 1–2, 1945.
9. Tranter, C. J. Integral Transforms. *John Wiley & Sons, Inc.*, p. 96, 1962.
10. Whittakar, E. T. and Watson, G. N. A Course of Modern Analysis. *Cambridge University Press*, 331, 320, 1963.

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A note on certain bilinear and bilateral generating relations for the Laguerre polynomial*

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Abstract

The author presents here a systematic discussion on certain bilinear and bilateral generating relations associated with the generalized Leguerre polynomials $\{L_n^{(\alpha)}(x)\}$ defined by

$$(1-t)^{-1-\alpha} \exp\left(-\frac{xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)\, t^n.$$

It is observed that the four formulas proved recently by Jain [7] are only specialized or limiting forms of the results of Erdélyi [4], Meixner [8], and the author [12, 13, 14].

For the Laguerre polynomial

$$L_n^{(\alpha)}(z) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!},$$

Jain[7] has recently proved, in these Proceedings, one bilinear and three bilateral generating functions in the forms†

$$(1) \qquad \sum_{n=0}^{\infty} \frac{n!\,(1+b+k)_n}{(1+a)_n\,(1+b)_n} L_n^{(a)}(x)\, L_n^{(a)}(y)\, z^n$$

$$= (1-z)^{-1-b-k} \exp\left[-\frac{(x+y)z}{1-z}\right. \sum_{n=0}^{\infty} \frac{(1+b+k)_n}{n!\,(1+a)_n\,(1+b)_n} \left[\frac{xyz}{(1-z)^2}\right]^n$$

$$\cdot\ {}_1F_1\left[\begin{matrix} a-b-k; \\ 1+a+n; \end{matrix} \frac{xz}{1-z}\right] {}_1F_1\left[\begin{matrix} -k; \\ 1+b+n; \end{matrix} \frac{yz}{1-z}\right],$$

$$(2) \qquad \sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n,\ d; \\ 1+a; \end{matrix} x\right] L_n^{(b)}(y)\, z^n = (1-z)^{-1-b} \exp\left(-\frac{yz}{1-z}\right)$$

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†It may be of interest to note that the formulas (3) and (4) appear incorrectly on page 938 of Jain's paper referred to above.

$$\cdot\ {}_3\Phi_G^{(1)}\left[d, d, d, c, a-b\, ; c, 1+a, 1+a\, ; -\frac{xz}{1-z}, \frac{xz}{1-z}, \frac{xyz}{(1-z)^2} \right],$$

$$\sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix} -n, & d\, ; \\ 1+a\, ; \end{matrix}\ x \right] L_n^{(b-n)}(y)\, z^n = (1+z)^b \exp(-yz) \tag{3}$$

$$\cdot\ \Phi_1\left[d, -b\, ;\ 1+a\, ; \frac{xz}{1+z}, xyz \right]$$

and

$$\sum_{n=0}^{\infty} \frac{(-)^n\,(-b)_n}{(1+a)_n}\, L_n^{(a)}(x)\ {}_2F_1\left[\begin{matrix} -n, & d\ ; \\ 1+b-n\, ; \end{matrix}\ y \right] z^n \tag{4}$$

$$= (1+z)^b\ {}_3\Phi_M^{(2)}\left[g, d, -b, d\, ; g, 1+a, 1+a\, ; -yz, \frac{xz}{1+z}, xyz \right],$$

where ϕ_1 denotes one of Humbert's confluent hypergeometric functions in two arguments defined by means of (see, *e.g.*, [5], p. 225)

$$\Phi_1[\alpha, \beta\, ; \gamma\, ; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}\,(\beta)_m}{(\gamma)_{m+n}}\, \frac{x^m}{m\,!}\, \frac{y^n}{n\,!},$$

and

$${}_3\Phi_G^{(1)}[\alpha, \alpha, \alpha, \beta, \beta'\, ; \gamma, \gamma', \gamma'\, ; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_{m+n+p}\,(\beta)_m\,(\beta')_n}{(\gamma)_m\,(\gamma')_{n+p}}\, \frac{x^m}{m\,!}\, \frac{y^n}{n\,!}\, \frac{z_p}{p\,!},$$

$${}_3\Phi_M^{(2)}[\alpha, \beta, \beta', \beta\, ; \gamma, \gamma', \gamma'\, ; x, y, z]$$

$$= \sum_{m,n,p=0}^{\infty} \frac{(\alpha)_m\,(\beta)_{m+p}\,(\beta')_n}{(\gamma)_m\,(\gamma')_{r+p}}\, \frac{x^m}{m\,!}\, \frac{y^n}{n\,!}\, \frac{z^p}{p\,!}.$$

A simple analysis will reveal the fact that the confluent triple hypergeometric functions occurring on the right-hand sides of (2) and (4) are essentially the Humbert functions ϕ_1 and ϕ_2, where ϕ_2 is defined by [5, p. 225]

$$\Phi_2[\alpha, \beta\, ; \gamma\, ; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m\,(\beta)_m}{(\gamma)_{m+n}}\, \frac{x^m}{m\,!}\, \frac{y^n}{n\,!},$$

and consequently, the reduced forms of the formulas (2) and (4) are

(5)
$$\sum_{n=0}^{\infty} {}_2F_1\left[\begin{matrix}-n, d ; \\ 1+a ;\end{matrix}\; x\right] L_n^{(b)}(y)\, z^n$$

$$= (1-z)^{d-b-1}(1-z+xz)^{-d} \exp\left(-\frac{yz}{1-z}\right)$$

$$\cdot\ \Phi_1\left[d, a-b ; 1+a ; \frac{xz}{1-z+xz}, \frac{xyz}{(1-z)(1-z+xz)}\right]$$

and

(6)
$$\sum_{n=0}^{\infty} \frac{(-)^n(-b)_n}{(1+a)_n} L_n^{(b)}(x)\; {}_2F_1\left[\begin{matrix}-n,\ d ; \\ 1+b-n ;\end{matrix}\; y\right] z^n$$

$$= (1+z)^b (1+yz)^{-d}\, \Phi_2\left[-b, d ; 1+a ; \frac{xz}{1+z}, \frac{xyz}{1+yz}\right]$$

respectively.

The main purpose of the present note is to observe that the formulas (1) and (2) in much more general forms were proved many decades earlier by Erdélyi [4] and Meixner[8] and that all these results admit themselves of further elegant generalisation which we have derived elsewhere (*see* [12], [13] and [14]).

$$L_n^{(\alpha)}(z) = \binom{n+\alpha}{n}\, {}_1F_1\left[\begin{matrix}-n ; \\ 1+\alpha ;\end{matrix}\; z\right],$$

and the well-known Kummer transformation [5, p. 253]

$${}_1F_1\left[\begin{matrix}a ; \\ c ;\end{matrix}\; z\right] = \exp(z)\; {}_1F_1\left[\begin{matrix}c-a ; \\ c ;\end{matrix}\; -z\right],$$

the formula (1) assumes the desired form

(7)
$$\sum_{n=0}^{\infty} \frac{(b+k)_n}{n\,!}\; {}_1F_1\left[\begin{matrix}-n ; \\ a ;\end{matrix}\; x\right] {}_1F_1\left[\begin{matrix}-n ; \\ b ;\end{matrix}\; y\right] z^n = (1-z)^{-b-k} \sum_{n=0}^{\infty} \frac{(b+k)_n}{(a)_n\,(b)_n\, n\,!}$$

$$\cdot\ {}_1F_1\left[\begin{matrix}b+k+n ; \\ a+n ;\end{matrix}\; \frac{xz}{z-1}\right] {}_1F_1\left[\begin{matrix}b+k+n ; \\ b+n ;\end{matrix}\; \frac{yz}{z-1}\right] \left[\frac{xyz}{(1-z)^2}\right]^n.$$

Obviously, this is a special case of Erdélyi's formula [4, p. 344]

(8)
$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!}\; {}_1F_1\left[\begin{matrix}\rho-n ; \\ a ;\end{matrix}\; x\right] {}_1F_1\left[\begin{matrix}\sigma-n ; \\ b ;\end{matrix}\; y\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(a)_n\,(b)_n\, n\,!}\; \Phi_2\left[\rho, \lambda+n ; a+n ; x, \frac{xz}{z-1}\right]$$

$$\cdot\ \Phi_2\left[\sigma, \lambda + n;\ b + n;\ y, \frac{yz}{z-1}\right]\left[\frac{xyz}{(1-z)^2}\right]^n, \ |z| < 1,$$

when $\rho = \sigma = 0$ and $\lambda = b + k$, Erdélyi [4, p. 344] does state the special case $\rho = \sigma = 0$ of his formula (8) in the form*

$$(9) \qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n\, n!}{(a)_n\,(b)_n} L_n^{(a-1)}(x)\, L_n^{(b-1)}(y)\, z^n = (1-z)^{-\lambda}. \sum_{n=0}^{\infty} \frac{(\lambda)_n}{(a)_n\,(b)_n\, n!}$$

$$\cdot\ {}_1F_1\left[\begin{matrix}\lambda + n; \\ a + n;\end{matrix}\ \frac{xz}{z-1}\right] {}_1F_1\left[\begin{matrix}\lambda + n; \\ b + n;\end{matrix}\ \frac{yz}{z-1}\right]\left[\frac{xyz}{(1-z)^2}\right]^n, \ |z| < 1.$$

Next we notice that the confluent hypergeometric function on the right-hand side of (2) equals

$$\sum_{n=0}^{\infty} \frac{(d)_n}{n!\,(1+a)_n}\left[\frac{xyz}{(1+z)^2}\right]^n \sum_{r,s=0}^{\infty} \frac{(d+n)_{r+s}\,(a-b)_s}{r!\,s!\,(1+a+n)_s}\,(-)^r\left(\frac{xz}{1-z}\right)^{r+s}$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n}{n!\,(1+a)_n}\left[\frac{xyz}{(1-z)^2}\right]^n \sum_{r=0}^{\infty} \frac{(d+n)_r}{r!}\left(\frac{xz}{z-1}\right)^r \sum_{s=0}^{r} \frac{(-r)_s\,(a-b)_s}{s!\,(1+a+n)_s},$$

$$= \sum_{n=0}^{\infty} \frac{(d)_n}{n!\,(1+a)_n}\left[\frac{xyz}{(1-z)^2}\right]^n \sum_{r=0}^{\infty} \frac{(d+n)_r\,(1+b+n)_r}{r!\,(1+a+n)_r}\left(\frac{xz}{z-1}\right)^r,$$

by Vandermonde's theorem [1, p. 3], and the bilateral generating relation (2) or (5) can thus be put in the equivalent form

$$(10) \qquad \sum_{n=0}^{\infty} \frac{(b)_n}{n!}\ {}_2F_1\left[\begin{matrix}-n, d; \\ a;\end{matrix}\ x\right] {}_1F_1\left[\begin{matrix}-n; \\ b;\end{matrix}\ y\right] z^n$$

$$= (1-z)^{-b} \exp\left(\frac{yz}{z-1}\right) \sum_{n=0}^{\infty} \frac{(d)_n}{n!\,(a)_n}\ {}_2F_1\left[\begin{matrix}b + n, d + n; \\ a + n;\end{matrix}\ \frac{xz}{z-1}\right]\left[\frac{xyz}{(1-z)^2}\right]^n.$$

Since

$$\exp\left(\frac{yz}{z-1}\right) = {}_1F_1\left[\begin{matrix}b + n; \\ b + n;\end{matrix}\ \frac{yz}{z-1}\right],$$

it is easy to see that (10) follows rather immediately from Meixner's formula ([8], see also [5], p. 84)

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*See also the formula (27 on page 288 of Erdelyi *et al*, [5]. For an elementary derivation of (9) with a slightly different right-hand side see the formula (2·3) of Srivastava [12].

(11)
$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!}\ {}_2F_1\left[\begin{matrix} -n, d\,; \\ a\,; \end{matrix}\ x\right] {}_2F_1\left[\begin{matrix} -n, e\,; \\ b\,; \end{matrix}\ y\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n\,(d)_n\,(e)_n}{n\,!\,(a)_n\,(b)_n}\ {}_2F_1\left[\begin{matrix} \lambda+n, d+n\,; \\ a+n\,; \end{matrix}\ \frac{xz}{z-1}\right]$$

$$\cdot\ {}_2F_1\left[\begin{matrix} \lambda+n, e+n\,; \\ b+n\,; \end{matrix}\ \frac{yz}{z-1}\right] \left[\frac{xyz}{(1-z)^2}\right]^n, \quad |z|<1,$$

when y is replaced by y/e, $\lambda = b$, and $e \to \infty$.

The foregoing analysis when applied to the remaining formulas (8) and (4) will readily reduce them to their equivalent forms

(12)
$$\sum_{n=0}^{\infty} \frac{(b)_n}{n\,!}\ {}_2F_1\left[\begin{matrix} -n, d\,; \\ a\,; \end{matrix}\ x\right] {}_1F_1\left[\begin{matrix} -n\,; \\ 1-b-n\,; \end{matrix}\ y\right] z^n$$

$$= (1-z)^{-b} \exp(yz) \sum_{n=0}^{\infty} \frac{(d)_n}{(a)_n} \frac{(-xyz)^n}{n\,!}\ {}_2F_1\left[\begin{matrix} b, d+n\,; \\ a+n\,; \end{matrix}\ \frac{xz}{z-1}\right]$$

and

(13)
$$\sum_{n=0}^{\infty} \frac{(b)_n}{n\,!}\ {}_1F_1\left[\begin{matrix} -n\,; \\ a\,; \end{matrix}\ x\right] {}_2F_1\left[\begin{matrix} -n, \ d\ ; \\ 1-b-n\,; \end{matrix}\ y\right] z^n$$

$$= (1-z)^{-b} \sum_{n=0}^{\infty} \frac{(d)_n}{(a)_n} \frac{(-xyz)^n}{n\,!}\ {}_1F_1\left[\begin{matrix} b\,; \\ a+n\,; \end{matrix}\ \frac{xz}{z-1}\right] {}_1F_0\left[\begin{matrix} d+n : \\ -\,; \end{matrix}\ yz\right]$$

respectively.

Both (12) and (13) are evidently immediate consequences of our bilinear generating relation (3·3) in [12], *viz.*

(14)
$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n\,!}\ {}_{p+1}F_q\left[\begin{matrix} -n, a_1, \ldots, a_p\,; \\ b_1, \ldots, b_q\,; \end{matrix}\ x\right] {}_{r+1}F_{s+1}\left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_r\,; \\ 1-\lambda-n, \beta_1, \ldots, \beta_s\,; \end{matrix}\ y\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{\prod\limits_{j=1}^{p} (a_j)_n \ \prod\limits_{j=1}^{r} (\alpha_j)_n}{\prod\limits_{j=0}^{q} (b_j)_n \ \prod\limits_{j=1}^{s} (\beta_j)_n} \frac{(-xyz)^n}{n\,!}$$

$$\cdot\ {}_{p+1}F_q\left[\begin{matrix} \lambda, a_1+n, \ldots, a_p+n\,; \\ b_1+n, \ldots, b_q+n\,; \end{matrix}\ \frac{xz}{z-1}\right] {}_rF_s\left[\begin{matrix} \alpha_1+n, \ldots, \alpha_r+n\,; \\ \beta_1+n, \ldots, \beta_s+n\,; \end{matrix}\ yz\right],$$

valid for such values of $|x|$, $|y|$, and $|z|$ that the two sides have a meaning, provided that $p \leqslant q$ and $r \leqslant s+1$.

It may be of interest to remark that in the course of an attempt to give extensions of the well-known Hille-Hardy formula (see, *e.g.*, [6], p. 189)

$$(15) \qquad \sum_{n=0}^{\infty} \frac{n!}{\Gamma(a+n+1)} L_n^{(a)}(x)\; L_n^{(\alpha)}(y)\, z^n = (1-z)^{-1} \exp\left[-\frac{(x+y)z}{1-z}\right]$$

$$\cdot\ (xyz)^{-\frac{1}{2}a}\; I_a\left[\frac{2\sqrt{(xyz)}}{1-z}\right], \quad |z| < 1,$$

which follows readily from (9) when $a = b = \lambda$, we have recently applied the Laplace and the inverse Laplace transform techniques and the method of multidimensional mathematical induction in order to obtain several new and distinct bilinear and bilateral generating functions for certain classes of generalized hypergeometric polynomials. These results include (14) and the bilinear generating function

$$(16) \qquad \sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\; {}_{p+1}F_q\left[\begin{matrix} -n, a_1, \ldots, a_p; \\ b_1, \ldots, b_q; \end{matrix}\; x\right] {}_{r+1}F_s\left[\begin{matrix} -n, \alpha_1, \ldots, \alpha_r; \\ \beta_1, \ldots, \beta_s; \end{matrix}\; y\right] z^n$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n \prod_{j=1}^{p} (a_j)_n \prod_{j=1}^{r} (\alpha_j)_n}{n! \prod_{j=1}^{q} (b_j)_n \prod_{j=1}^{s} (\beta_j)_n} \left[\frac{xyz}{(1-z)^2}\right]^n$$

$$\cdot\ {}_{p+1}F_q\left[\begin{matrix} \lambda+n, a_1+n, \ldots, a_p+n; \\ b_1+n, \ldots, b_q+n; \end{matrix}\; \frac{xz}{z-1}\right] {}_{r+1}F_s\left[\begin{matrix} \lambda+n, \alpha_1+n, \ldots, \alpha_r+n; \\ \beta_1+n, \ldots, \beta_s+n; \end{matrix}\; \frac{yz}{z-1}\right]$$

which unify scores of hitherto scattered results in the theory of generalized hypergeometric polynomials (see [9], [10] and [15]). For instance, the special case $p = q = 1$ of (16) corresponds to the main formula (4), p. 1320 of Brafman [2] when $\lambda - b_1$, and in the limit as $x \to 0$ our formulas (14) and (16), as also (2·8) in [12], reduce to the results proved earlier by Chaundy [3, p. 62]. For $p=q=r=s=1$, our bilinear relations (16), and (3·4) and (3·5) in [12], are due to Meixner [8] who obtained them two decades ago by transforming the Pochhammer contour integral associated with Gauss's hypergeometric function.

We should like to conclude by mentioning that several extensions of the generating relations in [11] and [12] to hold for certain classes of general double hypergeometric functions appear in our subsequent works [13] and [14]. Indeed in terms of the Appell function [5, p. 224]

$$F_1[\alpha, \beta, \beta' ; \gamma ; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n} (\beta)_m (\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!} \frac{y^n}{n!}$$

we find it worthwhile to record the following very specialized, but nevertheless interesting, forms* of some of our results :

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1\left[\begin{matrix}-n, a ; \\ b ;\end{matrix}\; x\right] {}_2F_1\left[\begin{matrix}-n, \mu ; \\ 1-\lambda-n ;\end{matrix}\; y\right] z^n \tag{17}$$

$$= (1-z)^{-\lambda} (1-yz)^{-\mu} F_1\left[a, \lambda, \mu ; b ; \frac{xz}{z-1}, \frac{xyz}{yz-1}\right];$$

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!} {}_2F_1\left[\begin{matrix}\rho-n, d ; \\ a ;\end{matrix}\; x\right] {}_2F_1\left[\begin{matrix}\sigma-n, e ; \\ b ;\end{matrix}\; y\right] z^n \tag{18}$$

$$= (1-z)^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda)_n (d)_n (e)_n}{n! (a)_n (b)_n} F_1\left[d+n, \rho, \lambda+n ; a+n ; x, \frac{xz}{z-1}\right]$$

$$\cdot F_1\left[e+n, \sigma, \lambda+n ; b+n ; y, \frac{yz}{z-1}\right] \left[\frac{xyz}{(1-z)^2}\right]^n .$$

The last formula reduces, when $\rho = \sigma = 0$, to Meixner's bilinear generating function (11) which, in turn, yields several hitherto scattered results including, *e. g.*, (4·3) and (4·6) of Weisner [16, p- 1037].

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*Note that in view of (5) and (6), the results (17) and (18) do incorporate the four formulas (1) through (4) as their specialized or limiting cases.

References

1. Bailey, W. N. Generalized hypergeometric series. *Cambridge Math. Tract No.* 32, *Cambridge,* 1935 ; Reprinted by Stechert-Hafner, New York, 1964.
2. Brafman, F. An ultraspherical generating function. *Pacific J. Math.*, **7**, 1319-1323, 1957.
3. Chaundy. T. W. An extension of hypergeometric functions. *Quart. J. Math. Oxford Ser.*, **14**, 55-78, 1943.
4. Erdélyi, A. Transformation einer gewissen nach Produkten konfluenter hypergeometrischer Funktionen fortschreitenden Reihe. *Compositio Math.*, **6**, 336-347, 1939.
5. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. Higher transcendental functions. *McGraw-Hill, New York,* Vol 1, 1953.
6. Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G. Higher transcendental functions, *McGraw-Hill, New York,* Vol 2, 1953.
7. Jain, R. N. Bilinear generating relations involving Laguerre polynomials. *Proc. Nat. Acad. Sci. India, Sec. A,* **36**, 937-944, 1966.

8. Meixner, J. Umformung gewisser Reihen, deren Glieder Produkte hypergeometrischer Funktionen sind. *Deutsche Math.*, **6**, 341–349, 1942.

9. Srivastava, H. M. and Joshi, C. M. Certain double Whittaker transforms of generalized hypergeometric functions. *Yokohama Math., J.*, **15**, 17–32, 1967.

10. Srivastava, H. M. and Singhal, J. P. Double Meijer transformations of certain hypergeometric functions. *Proc. Cambridge Philos. Soc.*, **64**, 425–430, 1968.

11. Srivastava, H. M. An infinite summation formula associated with Appell's function F_2, *Proc. Chambridge Philos. Soc.*, **65**, in Press, 1969.

12. Srivastava, H. M. An extension of the Hille–Hardy formula. *Math. Comp.*, **23**, (1969), in press; *See* also *Amer. Math. Soc. Notices*, **15**, 634-635, 1968.

13. Srivastava, H. M. Certain formulas associated with generalized Rice polynomials (in course of publication).

14. Srivastava, H. M. Infinite series of certain products involving Appell's double hypergeometric functions. *Glasnik Mat. Ser.* III, **4**, in press, 1969.

15. Srivastava, H. M. Certain series involving products of Laguerre polynomials. *Proc. Indian Acad. Sci. Sect.* A, **69**, in press 1969.

16. Weisner, L. Group-theoretic origin of certain generating functions. *Pacific J. Math.*, **5**, 1033–1039, 1955.

Proc. Nat. Acad. Sci., India, 40(A), IV, 1970

Reactions of Ortho-esters of Germanium-reactions of ethyl Orthogermanate with acetic and Benzoic acids

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Reactions of ethyl orthogermanate with acetic and benzoic acids in different stoichiometric ratios have been studied.

In a number of recent publications from these laboratories, interesting and useful results have been described in the study of the reactions of the alkoxy derivatives of aluminium[1,2], silicon[3-5], titanium[6-8], zirconium[9,10] with carboxylic acids and anhydrides, (*e. g.*, acetic acid, acetic anhydride, benzoic, lactic, mandelic and salicylic acids) in non-aqueous medium. In view of the above, it was considered of interest to study the reactions of ethyl orthogermanate with acetic and benzoic acids.

Ethyl orthogermanate on being allowed to react with one mole of acetic acid in refluxing benzene gave the triethoxy monoacetate derivative in almost quantitative yield :

$$Ge(OC_2H_5)_4 + CH_3COOH \rightarrow (C_2H_5O)_3\,Ge(OOC{\cdot}CH_3) + C_2H_5OH$$

when the above reaction was carried out in 1 : 2 molar ratio, only 1·7 moles of alcohol could be fractionated out azeotropically and a basic acetate, $GeO_{1\cdot5}(OOCCH_3)_{0\cdot25}(OC_2H_5)_{0\cdot75}$ was isolated from the reaction mixture.

In order to avoid the type of decomposition observed above, the reactions of ethyl orthogermanate with acetic acid were carried out in cold in 1 : 2 and 1 : 4 molar ratios when the resulting products were found to have acetate : germanium ratio of 0·88 : 1 and 1.08 : 1 respectively. The former compound on being heated in a bath of 100-110° for about half an hour appeared to yield a derivative with acetate : germanium ratio of 0·66 : 1.

The reaction between ethyl orthogermanate and acetic anhydride (1 : 4) in refluxing benzene yielded a product which was found to have acetate : germanium ratio of 0·89 : 1.

It appears that the main straight-forward reaction of ethyl orthogermanate with acetic acid or anhydried can be represented by the following equations :

$$Ge(OC_2H_5)_4 + CH_3COOH \rightarrow (C_2H_5O)_3\,Ge(OOCCH_3) + C_2H_5OH$$

$$Ge(OC_2H_5)_4 + (CH_3CO)_2O \rightarrow (C_2H_5O)_3\,Ge(OOCCH_3) + CH_3COOC_2H_5.$$

Further reactions are slow and accompanied with side decomposition.

It is interesting to recall that in the corresponding reactions of alkyl orthotitanates[8] with acetic acid or anhydried, a diacetate derivative is formed by an exothermic facile reaction and in this case also further reaction is much slower and accompanied by side reaction which mainly consist in the elimination of ethyl acetate :

$$Ti(OC_2H_5)_4 + 2(CH_3CO)_2O \xrightarrow{\text{fast}} (C_2H_5O)_2\,Ti(OOCCH_3)_2 + 2CH_3COOC_2H_5$$

$$(C_2H_5O)_2\,Ti(OOCCH_3)_2 + (CH_3CO)_2O \xrightarrow{\text{slow}} (C_2H_5O)\,Ti(OOCCH_3)_3 + CH_3COOC_2H_5$$

$$(C_2H_5O)\,Ti(OOCCH_3)_3 \rightarrow O = Ti(OOCCH_3)_2 + CH_3COOC_2H_5$$

In view of the above interesting results the reaction of germanium tetraacetate with ethanol (1 : 4) was also carried out at room temperature and the resulting product was shown to have acetate : germanium ratio of 1·05 : 1.

$$Ge(OOCCH_3)_4 + 3C_2H_5OH \rightarrow (C_2H_5O)_3\,Ge(OOCCH_3) + 3CH_3COOH \text{ (4 moles).}$$

The reactions between ethyl orthogermanate and benzoic acid have also been studied in different stoichiometric ratios. The reactions follow a pattern quite similar to the course of reactions, described by Mehrotra and coworkers for alkyl ortho-titanate :

$$Ge(OC_2H_5)_4 + C_6H_5COOH \rightarrow \underset{\text{I}}{(C_2H_5O)_3\,Ge(OOCC_6H_5)} + C_2H_5OH \qquad (1)$$

$$Ge(OC_2H_5)_4 + 2C_6H_5COOH \rightarrow \underset{\text{II}}{(C_2H_5O)_2\,Ge(OOCC_6H_5)_2} + 2C_2H_5OH \qquad (2)$$

$$Ge(OC_2H_5)_4 + 3C_6H_5COOH \rightarrow \underset{\text{III}}{(C_2H_5O)\,Ge(OOCC_6H_5)_3} + 3C_2H_5OH \qquad (3)$$

The preparation of germanium tetrabenzoate was also attempted by refluxing ethyl orthogermanate during the course of which only 2·87 moles of alcohol could be fractionated out azeotropically. On allowing the reaction mixture to reflux for a few more hours, a fine white solid separated. This was heated under reduced pressure to give a light amorphous powder which corresponded in analysis to germanyl benzoate. The reactions can, therefore, be represented by the following equations :

$$Ge(OC_2H_5)_4 + 3C_6H_5COOH \rightarrow (C_2H_5O)\,Ge(OOCC_6H_5)_3 + 3C_2H_5OH$$

$$(C_2H_5O)\,Ge(OOCC_6H_5)_3 \rightarrow (C_6H_5COO)_2\,Ge = O + C_6H_5COOC_2H_5 \qquad (4)$$

The formation of germanyl benzoate in the reaction (4) was confirmed by heating the compound (III) under reduced pressure :

$$(C_2H_5O)\,Ge(OOCC_6H_5)_3 \xrightarrow[\text{red. press.}]{\text{heat}} O = Ge(OOCC_6H_5)_2 + C_6H_5COOC_2H_5$$

Diethoxy germanium dibenzoate, described above (II), was found to interchange its ethoxy groups with higher alcohols, *e. g.* n-butanol and tertiary butanol :

$$(C_2H_5O)_2\,Ge(OOCC_6H_5)_2 + 2R'OH \rightarrow (R'O)_2\,Ge(OOCC_6H_5)_2 + 2C_2H_5OH$$

The alcohol produced in all the above reactions was removed azeotropically with benzene, and the course of reaction was followed both by estimating the liberated alcohol in the azeotrope and the analysis of germanium in the products.

Experimental

All glass apparatus with interchangeable joints was used and moisture was excluded from all the reactions.

Analytical procedures and methods for drying the reagents have been described elsewhere[11]. Ethyl orthogermanate was prepared by the ammonia method[12]. Germanium tetraacetate[13] was prepared by reacting germanium tetrachloride with acetic anhydride in the presence of thallous acetate. Acetic acid and anhydride were purified by fractionation. Benzoic acid (B. D. H., analar) was dried at 40°/1 5 mm immediately before use.

1. *Reaction between ethyl orthogermanate and acetic acid; molar ratio* 1 : 1.

Acetic acid (0·38g) was admitted to a mixture of ethyl orthogermanate (1·58g) and benzene (40g). On shaking the reaction mixture, a clear solution was obtained which was refluxed under the fractionating column for about two hours at 100°. The ethanol liberated in the reaction was fractionated off azeotropically. The excess of the solvent was removed under reduced pressure and the compound was dried at 32°/0·5 mm. for about two hours. A colourless liquid (1·60g), slightly more viscous than ethyl orthogermanate, fuming in moist air, was obtained.

Found : ethanol in the azeotrope, 0·26g (1 mole requires 0·28g).

%, found : Ge, 27·20 ; OC_2H_5, 50·63 ; $OOCCH_3$, 22·13 ; Calc. for

$(C_2H_5O)_3$ Ge $(OOCCH)_3$: Ge, 27·21 ; OC_2H_5, 50·66 ; $OOCCH_3$, 22·13.

2. *Reaction between ethyl orthogermanate and acetic acid; molar ratio* 1 : 2.

Acetic acid (1·14g) was treated with ethyl orthogermanate (2·82g) in refluxing benzene (40g) for about three hours, and the benzene-ethanol azeotrope was collected very slowly. More benzene (20g) was added and the mixture was refluxed for another two hours and the collection of the azeotrope was continued. The remaining solution was evaporated to dryness under reduced pressure. A colourless semi-solid miscible with benzene together with some insoluble impurity was obtained. The soluble portion was extracted with benzene. On drying it under reduced pressure a spongy white solid (1·09g), soluble in benzene, was obtained.

Found : ethanol in the azeotrope, 0·89g (2 moles require 1·027g).

%, found : Ge, 49·91 ; OC_2H_5, 23·20 ; $OOCCH_3$, 9·93 ; Calc. for

$GeO_{1·5}(OC_2H_5)_{0·75}(OOCCH_3)_{0·25}$: Ge, 50·01; OC_2H_5, 23·28; $OOCCH_3$, 10·17.

3. *Reaction between ethyl orthogermanate and acetic acid in cold; molar ratio* 1 : 2.

Acetic acid (0·99g) was introduced to a mixture of ethyl orthogermanate (2·06g) and benzene (2g). On shaking the reactants for some time a slight turbidity appeared which, however, redissolved on keeping the reaction mixture overnight. The solvent was removed under reduced pressure and the product was dried at 32°/0·5 mm. for about two hours. A colourless liquid (1·91g) was obtained.

%, found : Ge, 28·40 ; OC_2H_5, 50·40 ; $OOCCH_3$; 20·50.

(Ge : OC_2H_5 ratio = 1 : 2·86 ; Ge : $OOCCH_3$ ratio = 1 : 0·88).

The above compound (1·54g) was heated in a bath of 100–110° for about 30 minutes and then evacuated under reduced pressure for about two hours. A colourless liquid (1·27g) was obtained.

%, found : Ge, 30·58 ; OC_2H_5, 49·00 ; $OOCCH_3$, 16·50.

(Ge : OC_2H_5 ratio = 1 : 2·58 ; Ge : $OOCCH_3$ ratio = 1 : 0·66).

4. *Reaction between ethyl orthogermanate and acetic acid in cold* ; *molar ratio* 1 : 4.

A slight turbidity appeared on shaking a mixture of ethly orthogermanate (1·65g), acetic acid (1·57g) and benzene (2g). The mixture was kept overnight and then evacuated under reduced pressure at 34°/0·5 mm. for about three hours. A colourless, fuming liquid (1·52g) was obtained.

%, found : Ge, 29·00 ; OC_2H_5, 48·00 ; $OOCCH_3$; 25·50.

(Ge : OC_2H_5 ratio = 1 : 2·67 ; Ge : $OOCCH_3$ ratio = 1 : 1·08).

5. *Reaction between ethyl orthogermanate and acetic anhydride* ; *molar ratio* 1 : 4.

Acetic anhydride (2·80g) was added dropwise to a solution of ethyl orthogermanate (1·73g) in benzene (40g). On shaking a slight turbidity was noticed which disappeared on keeping the reactants overnight. The reaction mixture was refluxed at about 100° for 3–4 hours. The bath temperature was then raised and the distillate coming at 80° was collected. About 25g. of the distillate was withdrawn and the rest of the solvent was removed under reduced pressure and the product was dried at 55–60°/0·5–1·0 mm for about two hours. A slight turbid liquid (1·40g) was obtained.

%, found : Ge, 28·60 ; OC_2H_5, 49·60 ; $OOCCH_3$, 20·80.

(Ge : OC_2H_5 ratio = 1 : 2·8 ; Ge : $OOCCH_3$ ratio = 1 : 0·89).

6. *Reaction between germanium tetra–acetate and ethanol in cold* ; *molar ratio* 1 : 4.

Noticeable heat was produced when a mixture of germanium tetra–acetate (1·06g), ethanol (0·63g) and benzene (20g) was shaken together. The excess of the solvent was removed under reduced pressure and the compound was dried at 32°/1 mm. for about two hours. A colourless moblile liquid (0·76g) was obtained.

%, found : Ge, 26·00 ; OC_2H_5, 46·00 ; $OOCCH_3$, 22·20.

(Ge : OC_2H_5 ratio = 1 : 2·85 ; Ge : $OOCCH_3$ ratio = 1 : 1·05).

Thus in experiments (3–6) the final product was mainly, $(C_2H_5O)_3$ Ge$(OOCCH_3)$, which requires Ge, 27·21; OC_2H_5, 50·66 ; $(OOCCH_3$, 22·13%.

7. *Reaction between ethyl orthogermanate and benzoic acid* ; *molar ratio* 1 : 1.

Benzoic acid (0·97g), ethyl orthogermanate (2·02g) and benzene (40g) were shaken together. The clear solution, thus obtained, was refluxed under the fractionating column at 100–150° for about 1·5 hours. The ethanol liberated in the reaction was completely fractionated off azeotropically. The reaction maxture was rendered free of solvent under reduced pressure and the residual product was dried at 50°/0·5 mm. for 1·5 hours. A whitish slightly viscous liquid (2·58g) was obtained.

Found : ethanol in the azeotrope ; 0·36g (1 mole requires 0·368g).

%, found : Ge, 21·90 ; Calc. for $(C_2H_5O)_3$ Ge $(OOCC_6H_5)$: Ge, 22·07.

On heating 1·5g of the compound under reduced pressure, a colourless liquid (0·8g) b.p. 75–80°/3–4 mm. was obtained.

%, found : Ge, 15·10.

The residue in the flask contained Ge, 43·64%.

8. *Reaction between ethyl orthogermanate and benzoic acid ; molar ratio* 1 : 2.

Benzoic acid (1·87g) was admitted to a solution of ethyl orthogermanate (1·94g) in benzene (45g). On shaking a colourless solution was obtained which was allowed to reflux for about two hours at 110°. The distillate was withdrawn dropwise between 68–80°. The pure benzene (b. p. 80°) was distilled under high reflux ratio. The product was freed of solvent under reduced pressure at 50°/1 mm. A whitish viscous liquid (2·99g), miscible with benzene, was obtained.

Found ; ethanol in the azeotrope, 0·70g (2 moles require 0·70g).

%, found : Ge, 17·90 ; Calc. for $(C_2H_5O)_2\ Ge(OOCC_6H_5)_2$: Ge, 17·92.

9. *Reaction between* $(C_2H_5O)_2\ Ge(OOCC_6H_5)_2$ *and excess normal butanol* :

The compound, (from experiment, 8), (1·80g), normal butanol (3·0g) and benzene (30g), were refluxed for about two hours at 110°. The ethanol liberated in the reaction was fractionated out very slowly. The remaining solvent and butanol were removed under reduced pressure and the compound was dried at 40°/1 mm. for about two hours. A white semi-solid (2·00g), soluble in benzene, was obtained.

%, found : Ge, 15·69 ; Calc. for $(C_4H_9O^n)_2\ Ge(OOCC_6H_5)_2$: Ge, 15·74.

10. *Reaction between* $(C_2H_5O)_2\ Ge(OOCC_2H_5)_2$ *and excess tert. butanol* :

The compound (from experiment, 2), (1·82g), tertiary butanol (4·45g) and benzene (40g), were allowed to reflux at 115–20° as usual. The ethanol-benzene azeotrope was slowly collected. The volatile fractions were evaporated under reduced pressure and the compound was dried at 45°/2 mm. for about two hours. A white semi-solid (1·98g) miscible with benzene, was obtained.

Found : ethanol in the azeotrope, 0·38g (2 moles require 0·41g).

% found : Ge, 15·80 ; Calc. for $(C_4H_9{}^tO)_2\ Ge\ (OOCC_6H_5)_2$: Ge, 15·74.

11. *Reaction between ethyl orthogermanate and benzoic acid ; molar ratio* 1 : 3.

Benzoic acid (3·46g) was added with the help of benzene (50g) to ethly orthogermanate (2·39g) and the reaction mixture was then refluxed for about three hours at a bath temperature of 120°. The distillate was collected dropwise at 68°. The temperature of the distillate then rose to 76° after nearly 8 cc. of it were withdrawn. At this stage the reaction mixture was allowed to reflux more, for about two hours. The azeotrope was again collected dropwise till the temperature of distillate became steady at 80° and showed no tendency to decrease. The subsequent fractions of the distillate were collected separately under a high reflux ratio, 1 : 20. The product was freed of solvent under reduced pressure at 52°/1 mm. A white semi-solid (4·31g), soluble in benzene was obtained.

Found : ethanol in the azeotrope (fractions collected other than those distilling at 80°), 1·25g (3 moles require 1·30g).

%, found : Ge, 15·17 ; Calc. for $(C_2H_5O)\ Ge\ (OOCC_6H_5)_3$: Ge, 15·09.

Above compound (2·0g) was heated under reduced pressure at 100–150°/0·05 mm. for 2·5 hours. A white powder (1·28g) insoluble in benzene, was obtained.

%, found : Ge, 21·01 ; Calc. for $O{=}Ge\ (OOCC_6H_5)_2$ Ge, 21·93.

12. *Reaction between ethyl orthogermanate and excess* (< 4 *moles*) *benzoic acid* :

Ethyl orthogermanate (2·06g), benzoic acid (4·48g) and benzene (60g) were refluxed under the column for many hours (6 hours). The ethanol formed was removed azeotropically. A white finely divided solid began to separate when nearly all the ethanol was fractionated off. At this stage, the reaction mixture was further refluxed for about eight hours when more solid separated. It was then allowed to stand overnight. The supernatant liquor was decanted off along with some solid mass. The residue was thoroughly washed with dry benzene and dried at 70°/0·1 mm. A white powder (2·60g) was obtained.

Found : ethanol in the azeotrope, 1·072g (3 moles require 1·12g).

The above solid (2·01g), was heated in a bath of 105°/0·2 mm. for about two hours when droplets of a colourless liquid (having fruity smell) appeared in the receiver. A white powder (1·71g), insoluble in benzene, was left in the distilling flask.

%, found : Ge, 21·09 ; Calc. for $[O{=}Ge(OOCC_6H_5)_2]_n$: Ge, 21·95.

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References

1. Pande, K. C. and Mehrotra, R. C. *Z. anorg. allg. Chem.*, **286**, 291, 1956.
2. Rai, A. K., Mehrotra, R. K. and Mehrotra, R. C. *J. prakt. Chem.*, **4**, 105, 1963.
3. Mehrotra, R. C. and Narain, R. P. (Personal Communication).
4. ——— *J. Ind. Chem., Soc.*, **39**, 855, 1962.
5. Mehrotra, R. C. and Pant, B. C. (Personal Communication).
6. Pande, K. C. and Mehrotra, B. C. *Z. anorg. allg. Chem.*, **290**, 96 ; **291**, 97, 1957.
7. Varma, I. D. and Mehrotra, R. C. *J. prakt. Chem.*, **8**, 235, 1959.
8. Varma, I. D. and Mehrotra, R. C. *J. prakt. Chem.*, **4**, 247, 1960.
9. Kapoor, R. N. *Ph.D. Thesis, Lucknow University*, 1959.
10. Kapoor, R. N. and Mehrotra, R. C. *J. Amer. Chem. Soc.*, **80**, 3569, 1958 ; **82**, 3495, 1960.
11. Mehrotra, R. C. and Chandra, G. *J. Ind. Chem. Soc.*, **39**, 235, 1962.
12. Bradley, D. C., Kay, L. and Wardlaw, W. *J. Chem. Soc.*, 4916, 1956.
13. Schmidt, H., Blohm, C. and Jander, G. *Angew. Chem.*, **A59**, 233, 1947.

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On Resultant Kernel Functions

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Abstract

In this paper, some rules connecting different classes of self-reciprocal functions have been stated for the generalised Hankel transform, which may be introduced by using a symmetrical Fourier kernel due to Roop Narain. Further, by making use of these rules and Hardy's Formula to give resultant of two kernels

$$K(x) = \int_0^\infty P_1(x)\, P_2(xy)\, dy,$$

where $P_1(x)$ and $P_2(x)$ are Fourier kernels, certain theorems have been established to investigate the self-reciprocity of functions in the generalised Hankel transform. Various particular cases have also been discussed.

1. The Hankel transform

$$g(x) = \int_0^\infty (xy)^{\frac{1}{2}}\, J_\nu(xy)\, f(y)\, dy \tag{1·1}$$

may be generalised by using a symmetrical Fourier kernel, given by Narain[9] in the form :

$$g(x) = 2\beta\gamma \int_0^\infty (xy)^{\gamma-\frac{1}{2}}\, G^{q,\,p}_{2p,\,2q}\left[\beta^2(xy)^{2\gamma} \,\middle|\, \begin{matrix} a_1, \ldots, a_p, -a_1, \ldots, -a_p \\ b_1, \ldots, b_q, -b_1, \ldots, -b_q \end{matrix}\right] f(y)\, dy \tag{1·2}$$

where β and γ are real constants.

With $\beta = \frac{1}{2}$, $\gamma = 1$, $p = 0$, $q = 1$ and $b_1 = \nu/2$, (1·2) reduces to (1·1). Also, under certain substitutions [9], the kernel of (1·2) yields, as particular cases, various kernels, given by Watson [13], Bhatnagar [2], Narain { [6] and [7] } and Everitt [5].

In (1·2), if $f(x) \equiv g(x)$, then $f(x)$ will be said to be 'self-reciprocal' *i.e.* $R(a_p\, ;\, b_q)$ in the generalised Hankel transform (1·2).

The object of this paper is firstly to state some rules connecting different classes of self-reciprocal functions in (1·2) and to use these rules in establishing certain theorems which are useful in identifying the nature of resultant of two kernels as transforming a self-reciprocal function of known order into another of a different order. Some of these types of kernels have been obtained by making use of the theorems and various known and unknown results, which follow as particular cases, have also been discussed.

In the present discussion, for the sake of brevity, the notation of the type ${}_m\lambda_n$ has been used to stand for a set of parameters $\lambda_m, \lambda_{m+1}, \ldots, \lambda_n$.

2. *Rule I* : If $f(x)$ is $R(a_p\,;\, b_q)$ and belongs to $A(\alpha, a)$ [12, p. 252 § 9·7], then

$$g(x) = (1/x) \int_0^\infty P(y/x)\, f(y)\, dy$$

is $R(c_{p'}\,;\, d_{q'})$, provided that

$$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\pi_{j=1}^{q}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} + b_j + \frac{s}{2\gamma}\right) \pi_{j=1}^{q'}\, \Gamma\left(\frac{2\gamma+1}{4\gamma} + d_j - \frac{s}{2\gamma}\right)}{\pi_{j=1}^{p}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} - a_j + \frac{s}{2\gamma}\right) \pi_{j=1}^{p'}\, \Gamma\left(\frac{2\gamma+1}{2\gamma} - c_j - \frac{s}{2\gamma}\right)} \cdot \chi(s)\, x^{-s}\, ds,$$

where $\chi(s)$ is regular and satisfies the condition

(2·1) $$\chi(s) = \chi(1-s)\,, \qquad s = \sigma + it$$

in the strip

(2·2) $$a < \sigma < 1 - a$$

and

(2·3) $$\chi(s) = 0\left(e^{[(\frac{1}{2}q + q' - p - p')\pi/\gamma - \alpha + \eta]\,|\,t\,|}\right)$$

for every positive η and uniformly in any strip interior to (2·2) and c is any value of σ in (2·2)

In other words, $P(x)$ is a kernel transforming an $R(a_p\,;\, b_q)$ into an $R(c_{p'};\, d_{q'})$.

Rule II : If $f(x)$ is $R(a_p\,;\, b_q)$ and belongs to $A(\alpha, a)$ [12, p. 252 § 9·7], then the function

$$g(x) = \int_0^\infty P(xy)\, f(y)\, dy$$

is $R(c_{p'}\,;\, d_{q'})$, provided that

(2·4) $$P(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \beta^{-s}/\gamma\, \frac{\pi_{j=1}^{q}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} + b_j + \frac{s}{2\gamma}\right) \pi_{j=1}^{q'}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} + d_j + \frac{s}{2\gamma}\right)}{\pi_{j=1}^{p}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} - a_j + \frac{s}{2\gamma}\right) \pi_{j=1}^{p'}\, \Gamma\left(\frac{2\gamma-1}{4\gamma} - c_j + \frac{s}{2\gamma}\right)} . \chi(s)\, x^{-s} ds,$$

where $\chi(s)$ satisfies the conditions (2·1) and (2·3). Since the result is symmetrical, the kernel (2·4) also effects the converse transformation.

In other words, $P(x)$ is a kernel transforming an $R(a_p\,;\, b_q)$ into an $R(c_{p'};\, d_{q'})$ and *vice versa.*

The proofs of the above rules can be developed on the same lines as in the proofs of the corresponding theorems of the Hankel transform [12, pp. 268–270].

3. *Theorem 1* : The resultant

$$K(x) = \int_0^\infty P_1(y)\, P_2(xy)\, dy, \tag{3·1}$$

of the kernels of the form :

$$P_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{q_1} \Gamma\left(\frac{2\gamma+1}{4\gamma}+d_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p_1} \Gamma\left(\frac{2\gamma+1}{4\gamma}-c_j-\frac{s}{2\gamma}\right)} \cdot \chi(s)\, x^{-s}\, ds \tag{3·2}$$

and

$$P_2(x)=\frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \beta^{-s/\gamma}. \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{q_2} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p_2} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)} \omega(s)\, x^{-s} ds, \tag{3·3}$$

where $0 < c < 1,\ 0 < c' < 1$ and

$$\chi(s) = \chi(1-s)\ ; \qquad \omega(s) = \omega(1-s), \tag{3·4}$$

is a kernel transforming an $R(c_{p_1}\ ;\ d_{q_1})$ into an $R(\alpha_{p_2}\ ;\ \beta_{q_2})$ and *vice-versa.*

Proof : Putting for $P_2(xy)$ in (3·1) from (3·3) and then changing the order of integration, the resultant kernel becomes as

$$K(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \beta^{-s/\gamma}. \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{q_2} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p_2} \Gamma\left(\frac{2\gamma-1}{4\gamma}-\alpha_j+\frac{s}{2\gamma}\right)}. \tag{3·5}$$

$$\omega(s)\, x^{-s}\, ds \int_0^\infty y^{-s}\, P_1(y)\, dy.$$

The value of y–integral in (3·5) can easily be deduced from (3·2) by using Mellin's inversion formula and therefore substituting it in (3·5), we get

$$K(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \beta^{-s/\gamma} \frac{\prod_{j=1}^{q_1} \Gamma\left(\frac{2\gamma-1}{4\gamma}+d_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{q_2} \Gamma\left(\frac{2\gamma-1}{4\gamma}+\beta_j+\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p_1} \Gamma\left(\frac{2\gamma-1}{4\gamma}-c_j+\frac{s}{2\gamma}\right) \prod_{j=1}^{p_2} \Gamma\left(\frac{2\gamma-1}{2\gamma}-\alpha_j+\frac{s}{2\gamma}\right)} \theta(s)\, x^{-s}\, ds \tag{3·6}$$

where

$$\theta(s)=\frac{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma+1}{4\gamma}+{}_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma+1}{4\gamma}-a_j-\frac{s}{2\gamma}\right)}\cdot\omega(s)\,\chi(1-s),$$

which clearly satisfies the functional equation

$$\theta(s)=\theta(1-s).$$

Hence, by virtue of Rule II, given in section 2, it is obvious from the form of (3·6) that $K(x)$, defined by (3·1), is a kernel which transforms an $R(\alpha_{p_2}\ ;\beta_{q_2})$ into an $R(c_{p_1}\ ;d_{q_2})$ and *vice-versa.*

It only remains to justify the change in the order of integration in (3·5). Now, putting $s=c'+it$ and using

$$|\Gamma(A+\tfrac{1}{2}\,it/\gamma)|\approx\left(e^{-\frac{\pi}{4\gamma}|t|}\,|t|^{A-\frac{1}{2}}\right),\ \text{as}\ |t|\to\infty$$

in the integrand of (3·6), we see that its modulus is less than a constant multiple of

$$(3\cdot7)\qquad(\beta^{\gamma}x)^{-c'}\int_{-\infty}^{\infty}e^{-\frac{\pi}{4\gamma}(q+q_2-p-p_2)|t|}\,|t|^{\sum_1^p a_j+\sum_1^q b_j+\sum_1^{p_2}\alpha_j+\sum_1^{q_2}\beta_j+\left(\frac{2c'-1}{4\gamma}\right)(q+q_2-p-p_2)}\,|\omega(c'+it)|\,dt\int_0^{\infty}|P_1(y)|\,y^{-c'}\,dy.$$

The change of the order of integration will be permissible if both the y and t- integrals in (3·7) exist. These will exist if both the functions $P_1(x)$ and $P_2(x)$ belong to $A(\alpha,a)$ [12, p. 252 § 9·7]

Now, the modulus of the integrand of (3·3) is less than a constant multiple of

$$r^{-\sigma}e^{-(\alpha-\eta-|\theta|)|t|}\,|t|^{\sum_1^p\alpha_j+\sum_1^q b_j+\sum_1^{p_2}\alpha_j+\sum_1^{q_2}\beta_j+\left(\frac{2\sigma-1}{4\sigma}\right)(q+q_2-p-p_2)},$$

since

$$\omega(c'+it)=0\left(e^{\frac{\pi}{4\gamma}(q+q_2-p-p_2-\alpha+\eta)|t|}\right)$$

for every positive η and uniformly in any strip interior to $a<\sigma<1-a$, where c' is any value of σ in this strip. Thus the integral converges uniformly in any domain of x interior to the angle α and in any smaller angle *it* is $0(|x|^{-c'})$.

Since c' can be taken as near to $(1-a)$ or a as we please, $P_2(x)$ belongs to $A(\alpha, a)$. Similarly, $P_1(x)$ can be shown to belong to $A(\alpha, a)$.

This completes the proof of the theorem 1.

3·1. *Corollaries* : (*i*) With $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 1$, $q_1 = q_2 = q = 2$, $a_1 = k' - m' - \frac{1}{2} - \nu/2$, $b_1 = \nu/2$, $b_2 = \nu/2 + 2m'$, $c_1 = k - m - \frac{1}{2} - \mu/2$, $d_1 = \mu/2$, $d_2 = \mu/2 + 2m$, $\alpha_1 = l - n - \frac{1}{2} - \rho/2$, $\beta_1 = \rho/2$ and $\beta_2 = \rho/2 + 2n$, the above theorem reduces to a result, given by Saxena [10, p. 88] for $\chi_{\nu,k,m}$- transform.

(*ii*) On having $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 0$, $q_1 = q_2 = q = 1$, $b_1 = \nu/2$, $d_1 = \mu/2$ and $\beta_1 = \rho/2$, the theorem yields a known result due to Agrawal [1, p. 305], in the Hankel transform.

3·2. *Example* : Let us choose the kernels [11]

$$(3{\cdot}8) \qquad P_1(x) = x^{\gamma-\frac{1}{2}}\, G^{q+r,\, q_1+r}_{p+q_1+k,\, p_1+q+k}\left[x^{2\gamma} \middle| \begin{matrix} -{}_1d_{q_1},\ {}_1\sigma_k,\ -{}_1a_p \\ {}_1b_q,\ -{}_1\sigma_k,\ {}_1c_{p_1} \end{matrix} \right],$$

$$q + q_1 + 4r > p + p_1 + 2k > 2r,$$

which transforms an $R(a_p\,;\,b_q)$ into an $R(c_{p_1}\,;\,d_{q_1})$,

$$P_2(x) = x^{\gamma-\frac{1}{2}}\, G^{q+q_2+t,\, t}_{p+p_2+s,\, q+q_2+s}\left[\beta^2 x^{2\gamma} \middle| \begin{matrix} {}_1\rho_s,\ -{}_1a_p,\ -{}_1\gamma_{p_2} \\ {}_1b_q,\ {}_1\delta_{q_2},\ -{}_1\rho_s \end{matrix} \right],$$

$$2t < 2s + p + p_2 < 4t + q + q_2,$$

transforming an $R(a_p\,;\,b_q)$ into an $R(\gamma_{p_2}\,;\,\delta_{q_2})$ and *vice-versa.*

Substituting these kernels in (3·1), replacing $y^{2\gamma}$ by y and then evaluating the integral thus obtained with the help of a known result [4, p. 422(14)], we get the resultant kernel

$$(3{\cdot}8) \qquad K(x) = \tfrac{1}{2}\,(1/\gamma)\, x^{\gamma-\frac{1}{2}}\, G^{q+q_1+q_2+t+r,\; q+t+r}_{p+p_1+p_2+q+s+k,\; p+q+q_1+q_2+s+k}\left[\beta^2 x^{2\gamma}. \middle| \begin{matrix} -{}_1b_q,\ {}_1\sigma_r,\ {}_1\rho_s,\ -{}_1a_p,\ -{}_1\gamma_{p_2},\ r+{}_1\sigma_k,\ -{}_1c_{p_1} \\ {}_1d_{q_1},\ -{}_1\sigma_r,\ {}_1b_{q_2},\ {}_1\delta_{q_2},\ -{}_1\rho_s,\ -r+{}_1\sigma_k,\ {}_1a_p \end{matrix} \right],$$

as a kernel transforming an $R(c_{p_1}\,;\,d_{q_1})$ into an $R(\gamma_{p_2}\,;\,\delta_{q_2})$ and *vice-versa*, provided that $1 \leqslant q_1 + r \leqslant p + q_1 + k < p_1 + q + k < q + q_1 + q_2 + k - p_2$, $q + q_1 + 4r > p + p_1 + 2k > 2r$, $0 \leqslant 2t < p + p_2 + 2s < q + q_2 + 4t$, $\frac{1}{2}(p + p_1 + q - q_1) - r + k < q + r \leqslant p_1 + q + k$, $\frac{1}{2}(p + p_2 + q_2 + q) + s - t < q + q_2 + t \leqslant q + q_2 + s$, $|\arg \beta^2 x^{2\gamma}| < [\frac{1}{2}(q + q_2 - p - p_2) + 2t - s]\,\pi$, $Re(m_j + n_h) > -1$, $(j = 1, \ldots\ldots, q + r\,;\ h = 1, \ldots\ldots, q + q_2 + t)$ where $m_j = b_j$ $(j = 1, \ldots, q)$, $m_{q+j} = -\sigma_j$ $(j = 1, \ldots, r)$, $n_h = b_h$ $(h = 1, \ldots, q)$,

$n_{q+h} = \delta_h$ $(h = 1, \ldots, q_2)$, $n_{q+q_2+h} - \rho_h$ $(h = 1, \ldots, t)$ and $Re(r_j + s_h) < 1$ $(j = 1, \ldots, q_1 + r;\ h = 1, \ldots, t)$ where $r_j = -d_j$ $(j = 1, \ldots, q_1)$, $r_{q_1+j} = \sigma_j$ $(j = 1, \ldots, r)$, $s_h = \rho_h$ $(h = 1, \ldots, t)$.

3·2·1. *Particular cases* : (*i*) In (3·8), taking $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 1$, $q_1 = q_2 = q = 2$, $r = 0$, $t = 0$, $s = 0$, $k=0$, $b_1 = \nu/2$, $b_2=0$, $a_1=0$, $c_1 = k-m-\frac{1}{2}-\mu/2$, $d_1 = \mu/2$, $d_2 = 2m + \mu/2$, $\gamma_1 = l - n - \frac{1}{2} - \rho/2$, $\delta_1 = \rho/2$, $\delta_2 = 2n + \rho/2$ and using the identity [3, p. 209(7)] we arrive at a known kernel [10, p. 94], transforming an $R_\mu(k, m)$ into an $R_\rho(l, n)$ and *vice versa*.

(*ii*) In (3·8), putting $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = 1$, $q_1 = q_2 = q = 2$, $r = 0$, $t = 0$, $s = 0$, $k = 0$, $b_1 = -1$, $a_1 = 0$, $c_1 = k - m - \frac{1}{2} - \mu/2$, $d_1=\mu/2$, $d_2 = 2m+\mu/2$, $\gamma_1 = l-n-\frac{1}{2}- \rho/2$, $\delta_1 = \rho/2$, $\delta_2 = 2n + \rho/2$ and applying the identities [3, p. 209(7) and p. 215(2)] we get a kernel transforming an $R_\mu(k, m)$ into an $R_\rho(l, n)$ and *vice-versa*.

(*iii*) With $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 0$, $q_1 = q_2 = q=1$, $r=0$, $t=0$, $s=0$, $k = 0$, $b_1 = -1$, $d_1 = \mu/2$, and $\delta_1 = \rho/2$, (3·8) yields a known kernel [8, p. 59] which transforms an R_μ into an R_ρ.

(*iv*) Similarly, setting the parameters suitably in (3·8), it is easy to deduce the kernels due to Narain [8, p. 60] and Saxena [10, p. 94] transforming an R_μ into an $R_{\mu-1}$ and an R_μ into an R_ρ respectively.

4. *Theorem* 2 : The resultant

$$K(x) = \int_0^\infty P_1(xy)\, P_2(y)\, dy, \tag{4·1}$$

of the two kernels of the form :

$$P_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \beta^{-s/\gamma}\, \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma} + b_j + \frac{s}{2\gamma}\right) \prod_{j=1}^{q_1} \Gamma\left(\frac{2\gamma-1}{4\gamma} + d_j + \frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma} - a_j + \frac{s}{2\gamma}\right) \prod_{j=1}^{p_1} \Gamma\left(\frac{2\gamma-1}{4\gamma} - c_j + \frac{s}{2\gamma}\right)} \,.\, \chi(s)\, x^{-s} ds$$

and

$$P_2(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'+i\infty} \beta^{-s/\gamma}\, \frac{\prod_{j=1}^{q} \Gamma\left(\frac{2\gamma-1}{4\gamma} + b_j + \frac{s}{2\gamma}\right) \prod_{j=1}^{q_2} \Gamma\left(\frac{2\gamma-1}{4\gamma} + \beta_j + \frac{s}{2\gamma}\right)}{\prod_{j=1}^{p} \Gamma\left(\frac{2\gamma-1}{4\gamma} - a_j + \frac{s}{2\gamma}\right) \prod_{j=1}^{p_2} \Gamma\left(\frac{2\gamma-1}{4\gamma} - \alpha_j + \frac{s}{2\gamma}\right)} \,.\, \omega(s)\, x^{-s} de,$$

where $0 < c < 1$, $0 < c' < 1$ and $\chi(s) = \chi(1-s)$; $\omega(s) = \omega(1-s)$, is a kernel transforming an $R(c_{p_1} ; d_{q_1})$ into an $R(a_{p_2} ; \beta_{q_2})$.

Its proof follows, if we proceed on similar lines as those of theorem 1

4·1. *Corollaries* : (*i*) Having $\beta = \frac{1}{2}, \gamma = 1, p_1 = p_2 = p = 1, q_1 = q_2 = q = 2, a_1 = k'-m'-\frac{1}{2}-\mu/2, b_1=\mu/2, b_2 = 2m'+\mu/2, c_1 = k-m\ \frac{1}{2}-\nu/2, d_1 = \nu/2, d_2 = 2m+\nu/2, \alpha_1 = l-n-\frac{1}{2}-\lambda/2, \beta_1 = \lambda/2$, and $\beta_2 = 2n + \lambda/2$, the above theorem reduces to a result, given by Saxena [10, p. 95].

(*ii*) With $\beta = \frac{1}{2}, \gamma = 1, p_1 = p_2 = p = 0, q_1 = q_2 = q = 1, b_1 = \mu/2, d_1 = \nu/2$ and $\beta_1 = \lambda/2$, the theorem yields a known result due to Agrawal [1, p. 308].

4·2. *Example* : Let us have the kernels [11]

$$P_1(x) = x^{\gamma-\frac{1}{2}}\, G^{q+q_1+r,r}_{p+p_1+k,q+q_1+k}\left[\beta^2x^{2\gamma}\,\middle|\,\begin{matrix} {}_1\sigma_k, -{}_1a_p, -{}_1c_{p_1} \\ {}_1b_q, {}_1\alpha_{q_1}, -{}_1\sigma_k \end{matrix}\right], \quad 2r<2k+p+p_1<4r+q+q_1,$$

transforming an $R(a_p\ ;\ b_q)$ into an $R(c_{p_1}\ ;\ d_{q_1})$ and *vice-versa* ;

and

$$P_2(x) = x^{\gamma-\frac{1}{2}}\, G^{q+q_2+t,t}_{p+p_2+s,q+q_2+s}\left[\beta^2x^{2\gamma}\,\middle|\,\begin{matrix} {}_1\rho_s, -{}_1a_p, -{}_1\gamma_{p_2} \\ {}_1b_q, {}_1\delta_{q_2}, -{}_1\rho_s \end{matrix}\right], \quad 2t<2s+p+p_2<4t+q+q_2,$$

transforming an $R(a_p\ ;\ b_q)$ into an $K(\gamma_{p_2}\ ;\ \delta_{q_2})$ and *vice-versa.*

Substituting in (4·1), replacing $y^{2\gamma}$ by y and then evaluating the integral thus obtained with the help of the known result [4, p. 422(14)] we get the resultant kernel

$$K(x) = \frac{x^{\gamma-\frac{1}{2}}}{2\gamma\beta^2}\, G^{q+q_1+r+t,y+q_2+r+t}_{p+p_1+q+q_2+k+s,p+p_2+q+q_1+k+s}\left[x^{2\gamma}\,\middle|\,\begin{matrix} -{}_1b_q, -{}_1\delta_{q_2}, {}_1\rho_t, {}_1\sigma_k, -{}_1a_p, -{}_1c_{p_1}, {}_{t+1}\rho_s \\ -{}_1\rho_t, {}_1b_q, {}_1d_{q_1}, -{}_1\sigma_k, -{}_{t+1}\rho_s, {}_1a_p, {}_1\gamma_{p_2} \end{matrix}\right], \tag{4·2}$$

transforming an $R(c_{p_1}\ ;\ d_{q_1})$ into an $R(\gamma_{p_2}\ ;\ \delta_{q_2})$, provided that $0 \leqslant t \leqslant p + p_2 + s < q + q_2 + s < p_2 - p_1 + q + q_1 + s$, $0 \leqslant 2r < 2k + p + p_1 < 4r + q_1 + q$, $2t < 2s + p + p_2 < 4t + q + q_1$, $\frac{1}{2}(p + p_2 + q + q_2) + s - t < q + q_2 + t \leqslant q + q_2 + s$, $\frac{1}{2}(p + p_1 + q + q_1) + k - r < q + q_1 + r \leqslant q + q_1 + k$, $|\arg \beta^2 x^{2\gamma}| < [\frac{1}{2}(q + q_1 - p - p_1) + 2r - k]\pi$; $R(m_j + n_h) > -1$ $(j = 1, \ldots, q + q_2 + t\ ;\ h = 1, \ldots, q + q_1 + r)$ where $m_j = b_j$ $(j = 1, \ldots, q)$, $m_{q+j} = \delta_j$ $(j = 1, \ldots, q_2)$, $m_{q+q_2+j} = -\rho_j$ $(j=1, \ldots, t)$, $n_h = b_h$ $(j=1, \ldots, q)$, $n_{q+h} = d_h$ $(h = 1, \ldots, q_1)$, $n_{q+q_1+h} = -\sigma_h$ $(h = 1, \ldots, r)$; $Re(\rho_j + \sigma_h) < 1$ $(j = 1, \ldots, t\ ;\ h = 1, \ldots, r)$

and $\sum_1^s Re\ \rho_j - \sum_1^p Re\ a_j - \sum_1^{p_2} Re\ \gamma_j - \sum_1^q Re\ b_j - \sum_1^{q_2} Re\ \delta_j + \sum_1^s Re\ \rho_j + \frac{1}{2}(q+q_2-p-p_2+1) > (q + q_2 - p - p_2)$. $Re(\sigma_j)$ $(j = 1, \ldots, r)$.

4·2·1. *Particular cases* : (*i*) In (4·2), setting $\beta = \frac{1}{2}, \gamma = 1, p_1 = p_2 = p = 1, q_1 = q_2 = q = 2, r = 0, t = 0, s = 0, k = 0, b_1 > \mu/2, b_2 = 0, a_1 = 0, c_1 = k-m-\frac{1}{2}-\nu/2,$

$d_1 = \nu/2$, $d_2 = 2m + \nu/2$, $\gamma_1 = l - n - \frac{1}{2} - \lambda/2$, $\delta_1 = \lambda/2$, $\delta_2 = 2n + \lambda/2$ and using a known identity [3, p. 209(7)], we get a known kernel [10, p. 98] transforming an $R_\nu(k, m)$ into an $R_\lambda(l, n)$.

(*ii*) In (4·2), putting $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 0$, $q_1 = q_2 = q = 1$, $t = 1$, $s = 2$, $r = 0$, $k = 0$, $a_1 = 0$, $c_1 = 0$, $\gamma_1 = 0$, $b_1 = \mu/2$, $d_1 = 1 + \nu/2$, $\delta_1 = \nu/2$, $\rho_1 = -\nu/2$, $\rho_2 = 1 + \nu/2$, we obtain a kernel [1, p. 309] transforming an $R_{\nu+2}$ to an R_ν.

(*iii*) In, (4·2), having $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 0$, $q_1 = q_2 = q = 1$, $r = 0$, $t = 0$, $s = 0$, $k = 0$, $a_1 = 0$, $c_1 = 0$, $\gamma_1 = 0$ and either ($b_1 = \mu/2$, $\delta_1 = \nu/2$, $d_1 = \nu/2$) or ($b_1 = \nu/2$, $\delta_1 = \mu/2$, $d_1 = \mu/2$), we arrive at the kernel [1, p. 310] transforming an R_ν or an R_μ into itself respectively.

5. *Theorem* 3 : The resultant

$$K(x) = \int_0^\infty P_1(y)\, p_2(xy)\, dy, \tag{5·1}$$

of the two kernels of the form :

$$P_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{q_1}\Gamma\left(\frac{2\gamma+1}{4\gamma}+d_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{p_1}\Gamma\left(\frac{2\gamma+1}{4\gamma}-c_j-\frac{s}{2\gamma}\right)} \cdot \chi(s)\, x^{-s}\, ds$$

and

$$P_2(x) = \frac{1}{2\pi i}\int_{c'-i\infty}^{c'-i\infty} \frac{\prod_{j=1}^{q}\Gamma\left(\frac{2\gamma-1}{4\gamma}+b_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{q_2}\Gamma\left(\frac{2\gamma+1}{4\gamma}+\beta_j-\frac{s}{2\gamma}\right)}{\prod_{j=1}^{p}\Gamma\left(\frac{2\gamma-1}{4\gamma}-a_j+\frac{s}{2\gamma}\right)\prod_{j=1}^{p_2}\Gamma\left(\frac{2\gamma+1}{4\gamma}-\alpha_j-\frac{s}{2\gamma}\right)} \cdot \omega(s)\, x^{-s}\, ds,$$

where $0 < c < 1$, $0 < c' < 1$ and $\chi(s) = \chi(1-s)$, $\omega(s) = \omega(1-s)$, is a kernel transforming an $R(c_{p_1} ; d_{q_1})$ into an $R(\alpha_{p_2} ; \beta_{q_2})$.

The proof of this theorem can be developed on the similar lines as those of the theorem 1.

5·1. *Corollaries* : (*i*) Setting $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 1$, $q_1 = q_2 = q = 2$, $a_1 = k' - m' - \frac{1}{2} - \nu'/2$, $b_1 = \nu'/2$, $b_2 = 2m' + \nu'/2$, $c_1 = l - n - \frac{1}{2} - \mu'/2$, $d_1 = \mu'/2$, $d_2 = 2n + \mu'/2$, $\alpha_1 = k - m - \frac{1}{2} - \rho/2$, $\beta_1 = \rho/2$, and $\beta_2 = 2m + \rho/2$, the above theorem reduces to a result, given by Saxena [10, p. 99].

(*ii*) With $\beta = \frac{1}{2}$, $\gamma = 1$, $p_1 = p_2 = p = 0$, $q_1 = q_2 = q = 1$, $b_1 = \nu'/2$, $d_1 = \mu'/2$ and $\beta_1 = \rho/2$, the theorem yields a known result due to Agarwal [1,p.309].

5·2. *Example* : Let us choose the kernels [11].

$$P_1(x)=x^{\gamma-\frac{1}{2}}\,G^{q+r,\,q_1+r}_{p+q_1+k,\,p_1+q+k}\left[x^{2\gamma}\,\middle|\,\begin{matrix}-{}_1d_{q_1}\,,\ {}_1\sigma_k,\ -{}_1a_p\\ {}_1b_q,\ -{}_1\sigma_k,\ {}_1c_{p_1}\end{matrix}\right],\ q+q_1+4r>p+p_1+2k>2r,$$

transforming an $R(a_p\ ;\ b_q)$ into an $R(c_{p_1}\ ;\ d_{q_1})$;

and

$$P_2(x)=x^{\gamma-\frac{1}{2}}\,G^{q+t,\,q_2+t}_{p+q_2+s,\,p_2+q+s}\left[x^{2\gamma}\,\middle|\,\begin{matrix}-{}_1\delta_{q_2}\,,\ {}_1\rho_s,\ -{}_1a_p\\ {}_1b_q,\ -{}_1\rho_s,\ -{}_1\gamma_{p_2}\end{matrix}\right],\ q+q_2+4t>p+p_2+2s>2t,$$

transforming an $R(a_p\ ;\ b_q)$ into an $R(\gamma_{p_2}\ ;\ \delta_{q_2})$.

Putting these in (5·1), replacing $y^{2\gamma}$ by y and then evaluating the integral thus obtained with the help of the known result [4, p. 422(14)] we obtain the resultant kernel

$$(5{\cdot}2)\quad K(x)=\frac{x^{\gamma-\frac{1}{2}}}{2\gamma}\,G^{q+q_1+r+t,\,q+q_2+r+t}_{p+p_1+q+q_2+k+s,\,p+p_2+q+q_1+k+s}$$
$$\left[x^{2\gamma}\,\middle|\,\begin{matrix}-{}_1b_q,\ {}_1\sigma_r,\ -{}_1\delta_{q_2}\,,\ {}_1\rho_s,\ -{}_1a_p,\ r+{}_1\sigma_k,\ -{}_1c_{p_1}\\ {}_1d_{q_1}\,,\ -{}_1\sigma_r,\ {}_1b_q,\ -{}_1\rho_s,\ {}_1\gamma_{p_2}\,,\ -r+{}_1\sigma_k,\ {}_1a_p\end{matrix}\right],$$

as transforming an $R(c_{p_1}\ ;d_{q_1})$ into an $R(\gamma_{p_2}\ ;\delta_{q_2})$, provided that

$1 \leqslant q_1 + r \leqslant p + q_1 + k < p_1 + q + k < q + q_1 - q_2 + p_2 + k,\ 0 \leqslant q_2 + t \leqslant p + q_2 + s,\ \ 2r < p + p_1 + 2k < q + q_1 + 4r,\ \ 2t < p + p_2 + 2s < q + q_2 + 4t,$ $\frac{1}{2}(p + p_1 + q - q_1) + k - r < q + r \leqslant p_1 + q + k,\ \frac{1}{2}(p + p_2 + q\ \ q_2) + s - t < q + t \leqslant p_2 + q + s,\ |\arg x^2\lambda| < [\frac{1}{2}(q + q_2 - p - p_2) + 2t - s]\pi$; $Re(m_j + n_h) > -1\ (j = 1, \ldots, q + r\ ;\ h = 1, \ldots, q + t)$ where $m_j = b_j\ (j = 1, \ldots, q)$, $m_{q+j} = -\sigma_j\ (j = 1, \ldots, r)$, $n_h = b_h\ (h = 1, \ldots, q)$, $n_{q+h} = -\rho_h\ (h = 1, \ldots, t)$; $Re(r_j + s_h) < 1\ (j = 1, \ldots, q_1 + r\ ;\ h = 1, \ldots, q_2 + t)$ where $r_j = -d_j$ $(j = 1, \ldots, q_1)$ $r_{q_1+j} = \sigma_j\ (j = 1, \ldots, r)$, $s_h = -\delta_h\ (h = 1, \ldots, q_2)$, $s_{q_2+h} = \rho_h$ $(h = 1, \ldots, t)$.

5·2·1. *Particular cases* : (*i*) In (5·2), having $\beta = \frac{1}{2}, \gamma = 1,\ p_1 = p_2 = p = 0,$ $q_1 = q_2 = q = 1, r = 1,\ k = 2,\ t = 0,\ s = 0, a_1 = 0, c_1 = 0,\ \gamma_1 = 0,\ b_1 = \mu/2,$ $d_1 = 1 + \nu/2, \delta_1 = \nu/2, \sigma_1 = -\nu/2$, and $\sigma_2 = 1 + \nu/2$, we again get the kernel given by Agrawal [1, p. 309], transforming an $R_{\nu+2}$ into an R_ν.

(*ii*) Similarly, by setting the parameters suitably, the kernel due to Agrawal [1, p. 310], transforming an R_ν or an R_μ into itself, can easily be deduced as particular cases of (5·2).

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References

1. Agrawal, R. P. On some new kernels and functions self-reciprocal in the Hankel transform. *Proc. Nat. Inst. Sci , India*, **13**(6), 305-318, 1947.
2. Bhatnagar, K. P. Certain theorems on self-reciprocal functions. *Bull. Cal. Math. Soc.*, **47**, 43-52, 1955.
3. Erdélyi, A. Higher transcedental functions, **Vol. I**, *McGraw-Hill, New York*, 1953.
4. ——— Tables of integral transforms, **Vol. II**, *McGraw-Hill, New York*, 1954.
5. Everitt, W. N. On a generalisation of Bessel functions and a resulting class of Fourier kernels. *Quart. J. Math. Oxford*, **10** (**II**), 270-279, 1959.
6. Narain, R. On a generalisation of Hankel transform and self-reciprocal functions. *Rendi. Sem. Math., Torino*, **16**, 269-300, 1956-57.
7. ——— A Fourier kernel. *Math. Zeitschr.*, **70**, 297-299, 1959.
8. ——— On certain kernel functions. *Ganita*, **10**, 57-64, 1959.
9. ——— The G-functions as unsymmetrical Fourier kernel I. *Proc. Amer. Math. Soc.*, **13**, 950-959, 1962.
10. Saxena, R. K. *Ph.D. Thesis, Vikram University*, 1964.
11. Sharma, O. P. *Ph.D. Thesis, Indore University*, 1968.
12. Titchmarsh, E. C. Introduction to the theory of Fourier integrals. *Oxford*, 1948.
13. Watson, G. N. Some self-reciprocal functions. *Quart. J. Math. Oxford*, **2** (**I**), 298-309, 1931.

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Application of generalized function method to the theory water waves

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Abstract

The generalized function method is utilized in solving an initial value problem of water waves generated in an inviscid, incompressible and homogeneous ocean by a harmonically oscillating pressure distribution acting on its free surface. It is substantiated that this new approach to the theory of water waves is very simple, rigorous and useful from the point of view of sufficient generality and applicability. An asymptotic analysis is carried out for finding both the steady state and transient solutions of the problem. A discussion of the waves motions is made with a special reference to their charecteristic features and the effects of surface tension on the wave phenomena.

1. Introduction

Water wave phenomena have received a great deal of attention by Stoker (1), Miles (2), Debnath (3-5), Debnath and Rosenblat (6), Lamb (7-8), Sen (9) and many others. Several classical methods have so far been employed in studying the linearized theory of wave motions in an inviscid, incompressible homogeneous fluid with a free surface due to local disturbances acting on and below the free surface of the fluid. An extensive use of classical Fourier series and transform techniques in these problems are well known and readily available in the existing literature (Wehausen and Laitone (10), Lamb (8), and Stoker (1)) on the subject.

In recent years, the author (3-5) has clearly pointed out certain inherent difficulties involved in the steady state water wave problems and the classical Fourier analysis used in solving them. These difficulties are essentially related to the existence, uniqueness and the real singularities of the inversion Fourier integral representation of the solution. In order to resolve these difficulties, Debnath (3) alone and in collaboration with Rosenblat (6) have investigated the steady state and the transient problem of water wave phenomena due to a harmonically oscillating pressure distribution of quite general character. This new treatment enables them to derive a mathematically unique and physically realistic solution in a simple, straightforward and elegant manner.

It thus appears that the water wave problems in various situations are readily and rigorously accessible to the generalized function treatment. This approach can not only eliminate the deficiencies of the classical Fourier transform analysis, but also give the solution of physical interest. Thus it seems to be extremely useful device from the point of view of sufficient generality and applicability. Another convincing point about this method is that there is neither any need for modification of the basic flow equations nor any justification of the limit operation suggested by Lamb and Sen.

This paper is devoted to study of water waves produced in an inviscid, incompressible, homogeneous deep ocean by a harmonically oscillating pressure distribution of general nature "switched on" at time $t = 0+$. The aim of this work is two-fold. In the first place, the study provides an interesting and useful application of the generalized functions to the water wave problems. Secondly, a unique solution of physical interest is obtained rigorously without any help of a radiation condition or an equivalent device. An asymptotic analysis is carried out to investigate the steady state and the transient solutions. A discussion of the wave motions is made with a special reference to their characteristic features and the effects of surface tension on the wave phenomena. Some consequences and implications of the results are presented.

2. Statement of the initial value problem

We consider a linearized two-dimensional transient wave motion of an inviscid, incompressible, homogeneous fluid of infinite depth. We take the origin of the rectangular cartesian coordinates (x, y, z) on the free surface of the fluid of uniform density ρ and surface tension T. In the undisturbed state, $x-y$ plane is assumed to be horizontal free surface and z-axis vertical positive upwards.

The unsteady wave motion is set up on the free surface of the fluid by an oscillating periodic pressure of fixed frequency ω in the form

$$\left.\begin{aligned} p(x, t) &= P\, p(x)\, e^{i\omega t}\, H(t), \; -a \leqslant x \leqslant a \\ &= 0 \qquad\qquad , \text{ outside} \end{aligned}\right\} \tag{2·1}$$

on the undisturbed free surface $z = 0$ of the fluid, where P is a constant, $p(x)$ is any arbitrary function of x and $H(t)$ is the Heaviside unit step function of time t.

In view of the fact that the flow is irrotational, there exists a wave potential $\phi(x, z\,;\,t)$ which is governed by the Laplace equation

$$\nabla^2\phi \equiv \phi_{xx} + \phi_{zz}, \; -\infty < z \leqslant 0, \; -\infty < x < \infty, \tag{2·2}$$

where suffixes stand for the partial differentiation.

The linearized dynamic and kinematic free surface conditions are given by

$$\phi_t + g\eta - \frac{T}{\rho}\,\eta_{xx} = -\frac{P}{\rho}\; p(x)\; e^{i\omega t}, \; z = 0, \; t > 0, \tag{2·3}$$

$$\eta_t = \phi_z, \; z = 0, \; t > 0, \tag{2·4}$$

where $\eta = \eta(x, t)$ denotes the vertical surface elevation and g is the gravitational acceleration.

The bottom boundary condition is given by

$$\phi_z \to 0 \text{ as } z \to -\infty, \; t > 0, \tag{2·5}$$

The initial conditions are

$$\left.\begin{aligned} \phi(x, z, 0) &= \eta(x, 0) = 0 \\ [\,\phi_t\,(x, 0\,;\,t)\,]_{t=0} &= -\frac{1}{\rho}\, p(x, 0) \end{aligned}\right\} \tag{2·6 a, b}$$

In addition, we shall treat $\phi(x, z\,;\,t)$ and $\eta(x, t)$ as generalized functions of x in the sense of Lighthill (11) so that their Fourier transform exists with respect to x.

3. Integral solutions

It is convenient to use the joint Laplace and the generalized Fourier transformations defined by the integral

$$\overline{\overset{\approx}{\phi}} = \overset{\approx}{\phi}\,(k, z\,;\,s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\,dk \int_{0}^{\infty} e^{-st}\,\phi(x, z\,;\,t)\,dt \qquad (3{\cdot}1)$$

The joint transform enables us to simplify the equations (2·2) - (2·5) and it turns out that the solution for the transform functions is

$$\overline{\tilde{\phi}}(k,\; z\,;\,s) = -\frac{P\,\bar{p}(k)\,s\,e^{|k|z}}{\rho\,(s - i\omega)\,(s^2+\alpha^2)} \qquad (3{\cdot}2)$$

$$\overline{\tilde{\eta}}\,(k, s) = -\frac{P\,p(k)\,|\,k\,|}{\rho(s - i\omega)\,(s^2+\alpha^2)} \qquad (3{\cdot}3)$$

where $\alpha^2 \equiv \alpha^2(k) = (g + \frac{T}{\rho}\,k^3)\,|\,k\,|,$ $\qquad (3{\cdot}4)$

The inversion theorem of the Laplace and the Fourier transform together with the Fultung theorem of the former yields

$$\phi(x, z\,;\,t) = \frac{P}{\rho\sqrt{2\pi}} \int_{-\infty}^{\infty} (i\omega \cos \alpha t - \alpha \sin \alpha t - i\omega\,e^{i\omega t})\,\frac{\bar{p}(k)\;e^{|k|z+ikx}}{(\alpha^2 - \omega^2)}\,dk, \qquad (3\;5)$$

$$\eta(x, t) = \frac{P}{\rho\sqrt{2\pi}}\int_{-\infty}^{\infty} |\,k\,|\,\bar{p}\,(k)\,(\alpha \cos \alpha t - i\omega \;\sin \alpha t - \alpha e^{i\omega t})\,\frac{e^{ikx}}{\alpha(\alpha^2-\omega^2)}\,dk\,, \qquad (3{\cdot}6)$$

Usually these integrals can not be evaluated exactly and it is necessary to resort to asymptotic techniques.

4. Asymptotic analysis

It would be sufficient for the examination of the transient wave motions to evaluate either the integral (3·5) or (3·6). We next turn our attention to the evaluation (3·6) by using formulae for the asymptotic development of the generalized Fourier transform due to Lighthill (11) and Jones (12). In order to facilitate the analysis, it is convenient to set $T = 0$ in (3·6) and to include a discussion about the effects of surface tension on the wave motions in a subsequent section.

Integral (3·6) can be written as

$$\eta(x, t) = \frac{P}{2g\rho\,\sqrt{2\pi}}\,(T - S)\,, \qquad (4{\cdot}1)$$

where S and T represent the steady state and the transient wave integrals respectively and are given by

$$S = e^{i\omega t} \int_{-\infty}^{\infty} \alpha(k)\, \bar{p}(k) \left(\frac{1}{\alpha + \omega} + \frac{1}{\alpha - \omega} \right) e^{ikx}\, dk, \tag{4·2}$$

$$T = \int_{-\infty}^{\infty} \alpha(k)\, \bar{p}(k) \left(\frac{e^{-i\alpha t}}{\alpha - \omega} + \frac{e^{i\alpha t}}{\alpha - \omega} \right) e^{ikx}\, dk, \tag{4·3}$$

where α is given by (3·4) with $T = 0$.

It may be noted that the significant contributions to the steady state integral S, as $|x| \to \infty$, come from the poles of the integrand of (4·2), that is, from the points where $\alpha \pm \omega = 0$. In fact, the poles are at $k = \pm \frac{\omega^2}{g}$. We evaluate S by using the result (11, pages 43 and 52) and it turns out that as $|x| \to \infty$

$$S \sim \pi i \; sgn\, x\; \psi\left(\frac{\omega^2}{S}\right) \left(e^{i\frac{\omega^2 x}{g}} - e^{-i\frac{\omega^2 x}{g}}\right) + 0\left(\frac{1}{|x|}\right), \tag{4·4}$$

where

$$\psi\left(\frac{\omega^2}{g}\right) = \frac{\alpha\left(\frac{\omega^2}{g}\right) \bar{p}\left(\frac{\omega^2}{g}\right)}{\left(\frac{d\alpha}{dk}\right)_{k = \omega^2/g}} \tag{4·5}$$

with $\bar{p}(k) = \bar{p}(-k)$ which corresponds to some simple cases of physical interest. However, this even-character of $\bar{p}(k)$ is not necessary in a general situation.

The method of stationary phase (Jone, 12) in conjunction with the result stated above can be used for the asymptotic evaluation of (4·3) for large t. It is then necessary to locate the stationary points of (4·3) which are roots of equation $\frac{d\alpha}{dk} = \pm \frac{x}{t}$ and $k = \pm k_1$ with $k_1 = \frac{gt^2}{4x^2}$. Thus the transient contribution to (4·3) is obtained in the form

$$T \sim \alpha(k_1)\, \bar{p}(k_1) \left\{ \frac{2\pi}{t\,|\alpha''(k_1)|} \right\}^{\frac{1}{2}} \left[\frac{e^{-i\{\alpha(k_1)t + k_1 x + \frac{\pi}{4}\}}}{\alpha(k_1) + \omega} = \frac{e^{i\{t\,\alpha(k_1) - k_1 x - \frac{\pi}{4}\}}}{\alpha(k_1) - \omega} \right] + 0\left(\frac{1}{t^2}\right), \tag{4·6}$$

It remains to calculate the contribution to T from its polar singularities which are the same as those of S. To evaluate the polar contribution to T, we write

$$T_{polar} = T_1 + T_2, \tag{4·7}$$

where T_1 and T_2 are given by

$$T_1 = \int_{-\infty}^{\infty} \frac{\alpha(k)\, \bar{p}(k)\, e^{-i(\alpha t - kx)} dk}{\alpha + \omega}, \tag{4·8}$$

$$T_2 = \int_{-\infty}^{\infty} \frac{\alpha(k)\, \bar{p}(k)\, e^{i(\alpha t + kx)}\, dk}{\alpha - \omega} \tag{4·9}$$

It may be noted that T_1 has no poles in the range of integration and hence contributes nothing. On the other hand, T_2 has two poles at $k = \pm\, t \frac{\omega^2}{g}$ and the residue-contribution can easily be obtained. Making reference to the technique developed by Debnath and Rosenblat (6) together with a change of variable k to $\alpha(k)$, it turns out that

$$T_{polar} = \pi\, i\, e^{i\omega t}\, \psi\left(\frac{\omega^2}{g}\right) \left(e^{i\frac{\omega^2}{g}x} + e^{-i\frac{\omega^2}{g}x}\right), \tag{4·10}$$

for large $t > 0$. The multiplicative factor *sgnt* is omitted since $t > 0$ is the interest.

The asymptotic solution for the veritical surface elevation $\eta(x, t)$ is obtained in the form

$$\eta(x, t) \sim \frac{2P\, i\, \omega^2 \pi\, \bar{p}\left(\frac{\omega^2}{g}\right)}{\rho\, g^2 \sqrt{2\pi}}\, e^{i\left(\omega t - \frac{\omega^2}{g}|x|\right)}$$
$$+ \frac{P t^2\, \bar{p}\left(\frac{g t^2}{4x^2}\right) \sqrt{g}\left[\frac{g t}{2x\omega} \cos\left(\frac{g t^2}{4x} - \frac{\pi}{4}\right) + i\, g \sin\left(\frac{g t^2}{4x} - \frac{\pi}{4}\right)\right]}{2\sqrt{2}\, \rho\, \omega\, x^{5/2}\left[\left(\frac{g t}{2x\omega}\right)^2 - 1\right]} + 0\left(\frac{1}{t^2}\right), \tag{4·11}$$

Remark : Solution (4·11) breaks down at $gt = 2x\omega$ which may be regarded as the critical point. A special device advanced by Debnath (13) may be employed to find a solution valid at the critical point. As the solution for large t is of most interest, no attention is given to this point.

5. Some pressure distribution of physical interest

The entire analysis has been carried out for an arbitrary function $p(x)$ involved in (2·1). It would be sufficient for the investigation of the dominant features of the unsteady wave motions to take some simple form of $p(x)$, such as,

(*a*) $p(x) = (a^2 - x^2)^{-\frac{1}{2}}$, (*b*) $p(x) = \frac{c^2}{c^2 + x^2}$, (*c*) $p(x) = 1,$ (*d*) $p(x) = d^{-x^2/x_0^2}$ and

(*e*) $p(x) = \delta(x),$

where $\delta(x)$ is the Dirac function of distribution.

The corresponding Fourier transform $\bar{p}(k)$ of these cases (*a*) – (*e*) are respectively given by

(*a*) $\bar{p}(k) = \sqrt{\frac{\pi}{2}}\, J_0(ak),$ (*b*) $\bar{p}(k) = \sqrt{\frac{2}{\pi}}\, c\, e^{-|k|c},$

(*c*) $\sqrt{\frac{2}{\pi}}\, \frac{\sin ak}{k}$ (*d*) $e^{-k^2 m^2},\ \left(m^2 = \frac{x_0^2}{4}\right)$ and

(*e*) $\frac{1}{\sqrt{2\pi}}$

The asymptotic solution for $\eta(x, t)$ associated with case (*d*) has the form

$$\eta(x, t) \sim \frac{\sqrt{2\pi}\, P\, i\omega^2}{\rho\, g^2}\ e^{-x_0^2/4}\ e^{i\left(\omega t - \frac{\omega^2}{g}|x|\right)}$$

$$+ \frac{P\, t^2 \exp\left\{-m^2\left(\frac{gt^2}{4x^2}\right)^2\right\}\left[\frac{gt}{2x\omega}\ \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right) + i \sin \frac{gt^2}{4x} - \frac{\pi}{4}\right]}{2\sqrt{2}\ \rho\ \omega\ x^{5/2}\left[\left(\frac{gt^2}{2x\omega}\right)^2 - 1\right]} \tag{5·1}$$

This solution clearly suggests that its transient component decays to zero very rapidly as $t \to \infty$ for fixed x. Consequently, an ultimate steady state is established in the limit. An exactly same conclusion can be drawn in all other pressure distributions confined over a finite region of the free surface. The nature of (5·1) in non-dimensional form at fixed time $t^* = 25\pi$ is exhibited by Figure 1. The non-dimensional quantities η^*, x^*, t^* are defined by $\eta^* = \frac{P\omega^4}{\rho g^3}\ \eta$, $t^* = \omega t$ and $x^* = \frac{\omega^2}{g}\ x.$

The asymptotic solution for $\eta(x, t)$ related to case (*e*) has the representation

$$\eta(x, t) \sim \frac{Pi\omega^2}{\rho g^2}\ e^{i\left(\omega t - \frac{\omega^2}{g}|x|\right)}$$

$$+ \frac{P\sqrt{g}\ t^2\left[\frac{gt}{2x\omega}\ \cos\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right) + i \sin\left(\frac{gt^2}{4x} - \frac{\pi}{4}\right)\right]}{4\sqrt{\pi}\ \rho\ \omega\ x^{5/2}\left[\left(\frac{gt}{2x\omega}\right)^2 - 1\right]} + 0\left(\frac{1}{t^2}\right), \tag{5·2}$$

It may be observed that this result follows from that of case (*c*) as a limit $P \to \infty$ $a \to 0$ provided $2Pa$ tends or a finite constant.

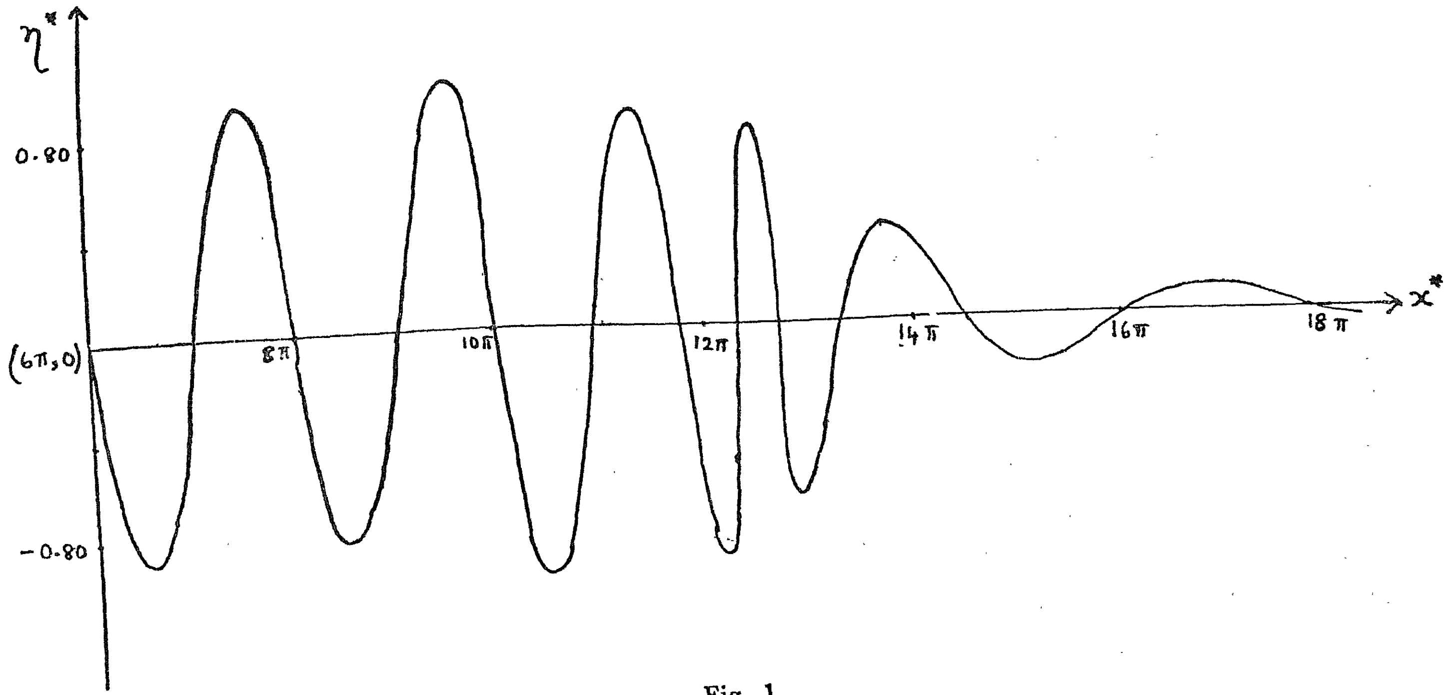

Fig. 1
Surface elevation in case 5(d) at fixed time $t = 25\,\pi$.

6. Conclusions

It may be inferred from the above asymptotic solution that the ultimate steady state is attained in the limit $t \to \infty$ for fixed x. The steady state solution for the surface elevation $\eta(x, t)$ has the form

$$\eta(x, t) \sim \frac{\sqrt{2\pi}\; Pi\omega^2}{\rho g^2}\, \bar{p}\left(\frac{\omega^2}{g}\right) e^{i\left(\omega t - \frac{\omega^2}{g}|x|\right)}, \tag{6·1}$$

This solution represents progressive surface waves propagating with the phase velocity $\frac{g}{\omega}$ and the group velocity $\frac{g}{2\omega}$, and the amplitude of the waves is constant.

This analysis shows that the present method of solution provides an interesting example of the application of generalized function method in the theory of water waves. Furthermore, a comparison of this method with the classical methods suggests that the use of generalized functions simplifies the analysis considerably.

7. Effects of surface tension

It has already been seen that surface tension enters into the water wave phenomena through equation (2·3). Consequently, the polar singularities of the wave integrals are at the points where

$$s^2 + \left(g + \frac{T}{\rho} k^2\right) |k| = 0, \tag{7·1}$$

Naturally, the stationary points related to the transient integrals suffer from a change due to inclusion of surface tension.

Without entering into mathematical details, it may be noted that real roots of (7·1) and the stationary points associated with the transient integrals are given by

$$k = k_0{}^* = \frac{\omega^2}{g} - \varepsilon,\ \varepsilon = \frac{T}{\rho}\left(\frac{\omega^3}{g^2}\right)^2 > 0, \tag{7·2}$$

$$k = k_1{}^* = \frac{gt^2}{4x^2}\left(1 + \frac{5T}{\rho}\right), \tag{7·3}$$

where surface tension T is very small.

Some conclusion can now be drawn concerning the role of surface tension. In the first place, both the steady state and the transient solutions are independently modified by the surface tension. The phase and the group velocities are $\frac{\omega}{k_0{}^*}$ and $\frac{\omega}{2k_0{}^*}$ respectively and evidently are greater than those of the corresponding results obtained without including the effects of surface tension. Secondly, it is of special interest to point out that the transient solution related to case 5(e) is $0(t^{-4})$ and hence tends to zero as $t \to \infty$ for fixed x. Consequently, the ultimate steady state is reached in the limit. Like viscosity, surface tension is really responsible for the attainment of the steady state.

Physically, the principal effects of surface tension is to increase the velocity of energy propagation and makes the energy more readily distributed among the rapidly travelling waves. It is thus highly likely that the intense accumulations of energy are no longer possible in the far field as $t \to \infty$. This is one of the most significant contrasts with the corresponding problem which takes on account of surface tension.

8. Concluding remarks

It appears from the above analysis that wave phenomena in fluids and elastic solids in various situations can readily and rigorously be investigated with the aid of the generalized function treatment. It may be interesting to analyze the above problem in a more general geometric configuration. Such study will be considered in a subsequent paper.

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References

1. Stoker, J. J. Water waves. *Interscience*, 1957.
2. Miles, J. W. Transient Gravity Wave Response to an Oscillating Pressure. *J. Fluid Mech.*, **13**, 145, 1962.
3. Debnath, L. A New Treatment of the Steady State Wave Problem I. *Proc. Nat. Inst. Sci.*, (*India*), A-**35**, 502, 1969.
4. ——— On Transient Development of Axisymmetric Waves. *Proc. Nat. Inst. Sci.*, (*India*), A-**35**, 567, 1969.
5. ——— An Asymptotic Treatment of the Transient Development of Axisymmetric Surface Waves. *Appl. Sci. Res.*, **21**, 24, 1969.
6. Debnath, L. and Rosenblat, S. The Ultimate Approach to the Steady State in the Generation of Waves on Running Stream. *Quart. Jour. Mech. and Appl. Math.* (*Oxford*), **22**, 221, 1969.
7. Lamb, H. Hydrodynamics. *Cambridge University Press*, 1932.
8. ——— On deep Water Waves. *Proc. Lond. Math Soc.*, **2**, 371, 1905.
9. Sen, A. R. Deep Water Surface Waves due to Arbitrary Periodic Pressure. *Proc. Nat. Inst. Sci.* (*India*), **28**-A, 612, 1962.
10. Wehausen, J. V. and Laitone, E. V. Handbook der Physik, IX, *Springer-Verlag*, 1960.
11. Lighthill, M. J. An Introduction to Fourier Analysis and Generalized Functions. *Cambridge University Press*, 1958.
12. Jones, D. S. Generalized Functions. *McGraw Hill*, 1966.
13. Debnath, L. On Transient Wave Motions in Fluid. *Ph.D. Thesis*, *University of London*, 1967.

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Kinetics and Mechanism of Reaction between Bromine and Quinoline in Carbon-Tetrachloride Solution.

Part II

By

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Abstract

The product ($C_9H_7NBr_2$) of the reaction between bromine and quinoline has been isolated and analysed for available bromine.

Calculated for : $C_9H_7NBr_2$: Br, 55·30%

Found : Br, 55·00%

The rate equation has been derived as

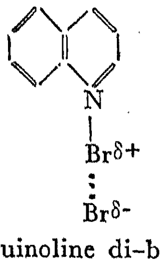
$$-\frac{d\,[Br_2]}{dt} = \frac{2k_1k_2\,[Qu]^2\,[Br_2]}{k_{-1} + k_2\,[Qu]}$$

Where k_{-1}, k_1 and k_2 are the velocity constant of the different steps of the reaction.

Introduction

In this paper we have studied the nature of the reaction mechanism and the product formed, when bromine and quinoline react in carbontetra-chloride solution. The experimental part has been published in the Indian J. of Chemistry in 1968, Vol. 6, No. 6, pp. 306–308.

The exact nature of the reaction product when bromine reacts with quinoline in carbontetra-chloride, at ordinary temperature has been discussed by Eisch[1,2] and the structure formula which has been ascribed to this addition product is as follows :



(quinoline di-bromide)

The reaction between bromine and quinoline is very fast and when the concentration of quinoline is comparable to that of bromine (*i.e.* for initial [bromine] = 15·52 × 10^{-3}M, [quinoline] = 67·58 × 10^{-3}M, 33·78 × 10^{-3}M and 16·89 × 10^{-3}M respectively) the total order of the reaction is three -two with respect to quinoline and one with respect to bromine. When the reaction is studied in the presence of excess of quinoline, (*i.e.* for initial [bromine] = 15·52 × 10^{-3}M, [quinoline] = 8·41 × 10^{-2}M, 12·62 × 10^{-2}M, 16·83 × 10^{-2}M and 50·50 × 10^{-2}M respectively and for initial [quinoline] = 42·08 × 10^{-2}M, [bromine] = 27·16 × 10^{-3}M, 23·28 × 10^{-3}M, 13·58 × 10^{-3}M and 11·64 × 10^{-3}M respectively) the reaction becomes very fast and also, rapid precipitation of quinoline di-bromide occurs. Under such conditions the reaction becomes second order—one with respect to each of the reactants.

Study of the Reaction Product

By equivalent method it is found that number of moles of bromine required by one mole of quinoline is one. Also the orange precipitate of the reaction product was collected, washed with fresh carbon-tetrachloride and dried in a desiccator over a mixture of KOH pellets and paraffin shavings. After the orange solid was dried in the dark, it was analysed for "available bromine". The prepared samples were treated with 15 ml. of 40% KI Solution and 15 ml. of 5 N acetic acid and the liberated iodine was titrated with standard sodium thiosulphate solution.

Calculated for : $C_9H_7NBr_2$: Br, 55·30%

Found : Br ; 55·0%

The orange adduct melted at 81-82°C. Treatment with sodium sulphite and ammonia solution regenerated quinoline. Upon standing in air, it slowly lost bromine vapour and developed yellow flakes.

Mechanism

We propose the following steps during the reaction between bromine and quinoline in carbon-tetrachloride solution.

$$\text{Qu.} + \text{Br}_2 \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} \text{Qu} \ldots . \text{Br}_2^* \qquad (1)$$

$$\text{Qu} \ldots \text{Br}_2^* + \text{Qu.} \xrightarrow{k_2} 2\ \text{Qu. Br} \qquad (2)$$

$$\text{Qu. Br} + \text{Br}_2 \xrightarrow{k_3} \text{Qu. Br}_2 + \text{Br} \qquad (3)$$

$$\text{Br} + \text{Br} \rightleftharpoons \text{Br}_2 \qquad (4)$$

The compound Qr . . . . Br_2^* may be regarded as activated complex in which the atoms have not, yet, been organized into a normal molecule, obviously the rate of variation of concentration is given by the expression.

$$-\frac{d\,[\text{Br}_2]}{dt} = k_1\,[\text{Qu}]\,[\text{Br}_2] + k_3\,[\text{Br}_2]\,[\text{Qu Br}] - k_{-1}\,[\text{Qu} \,.\,.\, \text{Br}_2^*] \qquad (5)$$

From principle of stationary state we have :

$$-\frac{[\text{Qr} \,.\,.\, \text{Br}_2^*]}{dt} = k_1\,[\text{Qu}]\,[\text{Br}_2] - k_{-1}\,[\text{Qu} \ldots \text{Br}_2^*] - k_2\,[\text{Qu} \,.\,.\, \text{Br}_2^*]\,[\text{Qu}] = 0$$

or $$[Qu \ldots Br_2^*] = \frac{k_1 [Qu] [Br_2]}{k_{-1} + k_2 [Qu]} \qquad (6)$$

and $$\frac{d [Qu\, Br]}{dt} = k_2 [Qu \ldots Br_2^*] [Qu] - k_3 [Qu\, Br] [Br_2] = 0$$

or $$[Qu\, Br] = \frac{k_2 [Qu \ldots Br_2^*] [Qu]}{k_3 [Br_2]} \qquad (7)$$

From Equs. (5), (6) and (7), we have

$$-\frac{d [Br_2]}{dt} = \frac{2k_1 k_2 [Qu]^2 [Br_2]}{k_{-1} + k_2 [Qu]} \qquad (8)$$

The reaction has been studied at the very low concentration of bromine (*i.e.* $15{\cdot}52 \times 10^{-3}$M) and hence if the $[Qu] = [Br_2]$ or the concentration of quinoline is slightly greater than the concentration of bromine, the values of $k_2 [Qu]$ and k_{-1} in the denominator of equ. 8. Can be neglected as the values are negligible and equ : (8) behaves as a third order expression—two with respect to quinoline and one with respect to bromine. When the concentration of quinoline is very large as compared to bromine so that $k_2 [Qu] >> k_{-1}$. The value of k_{-1} in denominator of equ. 8. Can be neglected, as it is negligible and the equ. (8). Can be written in the form.

$$-\frac{d [Br_2]}{dt} = \frac{2k_1 k_2 [Qu]^2 [Br_2]}{k_2 [Qu]} \qquad (9)$$

or $$-\frac{d [Br_2]}{dt} = 2k_1 [Qu] [Br_2] \qquad (10)$$

which is a second order expression, one with respect of each of the reactants.

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References

1. Eisch, J. J. *J. Org. Chem.*, **27**, 1318, 1962.
2. John. J. Eisch and Bruno. *J. Org. Chem.*, **28**, 2865, 1963.

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Effect of algae and phosphorus on the formation of aminoacids

By

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Abstract

Blue green algae Anabaena and Tolypothrix fix atmospheric nitrogen, along with carbon saving. The efficiency of nitrogen fixation by algae is more when the systems are phosphated with TATA or GERMAN basic slag. The amount and number of aminoacids is more in the systems which are inoculated with algae, more in light than in the dark.

Introduction

Since the discovery of phosphorus by Brandt[1] in 1969 has lead the scientist to make intensive studies, because of the unique and paramount importance of this element and its compounds to agriculture, biochemical processes and industry.

As in all living organisms, compounds containing phosphorus play important roles in nearly all phases of metabolism particularly in energy transformation reactions Krauss[2] has emphasised the need of phosphorus for the optimum growth of algae. Free amino acids are rapidly degraded in soil their amount, however, may quickly increase after available carbohydrates and inorganic nitrogen have been supplied. According to several workers[3,4] the turnover of protein in soil depends on the nature of the organic matter being degraded and the environmental conditions. Bremner[5] found however that organic matter tended to attain a constant protein and aminoacids composition. According to Watanabe[6], Fowden[7] and Williams and Burris[8], aminoacids like aspartic, glutamic acid, alanine, glycine, valine serine and phenyl alanine are present in all the blue green algae. Dhar and Roy[9] have also shown the formation of aminoacids by exposing normal solution of nitrate and glucose with titania. Simonart and Peeters[10] studied the aminoacids of forest soils, leaf moulds and green house soil and reported that aspartic acid, glutamic acid, serine and alanine occured in all samples. Recently Payne *et al*[11] noted that aqueous soil extract concentrated by freeze drying yielded ninhydrin positive spots on paper.

In the present investigation the role of algae and phosphorus on the formation of amino acids in soil has been studied.

Experimental

The experiments were carried out with the Institute soil. All the ingredients *viz* : soil, phosphates (Tata or German Basic Slag) and carbonaceous material (wheat straw) used in these experiments were sieved to 100 mesh. 200 gms of soil were taken in enamelled dishes. To this the required carbonaceous materials and phosphates were introduced as 1% carbon and 0·5% P_2O_5 respectively on an oven dry basis and the contents were thoroughly mixed in a big pestle and mortar and

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the whole mixture was made as uniform as possible. Then wherever required the soil was inoculated with algae *i. e.* Chlorella vulgaris, Anabaena naviculoides and Tolypothrix tenuis. Two similar sets were arranged side by side, one of which was exposed to light under an electric bulb and the other was covered with a thick black cloth. In all the dishes the moisture content was maintained at 40% level throughout the experiment and the contents were stirred daily. At regular intervals of time, composite samples were taken and were analysed for total carbon, total nitrogen, available P_2O_5 and aminoacids. The quantitative estimation of aminoacids was made by Harding and Mclean[12] procedure and qualitative detection by two dimensional paper chromatography. Bacteria free unialgal cultures of the above three algae were obtained from Dr. A. K. Mitra, an eminent algologist of the University of Allahabad.

The following abbreviations for the aminoacids studied have been used. Al for alanine, Va for valine, Asp for aspartic acid, Leu for leucine, Gly for Glycine, Glu for Glutamic acid, Ly for Lysine, Ar for Arginine and Hi for Histidine.

The experimental observation are recorded in the following pages.

Percentage analysis of the Ingredients

	Soil	Tata Basic slag	German Basic slag	Wheat straw
HCl insoluble	78·326	15·7485	11·3978	5·126
Sesquioxide	9·2478	–	–	1·4154
Fe_2O_3	4·167	15·4976	14·6893	0·6064
CaO	1·8765	38·6778	42·2897	0·8442
K_2O	1·1537	0·6337	traces	0·8036
MgO	1·5198	4·8247	4·9434	0·4078
Total P_2O_5	0·2108	7·468	17·2614	0·6038
Av. P_2O_5	0·0136	4·1024	9·8876	–
Total carbon	0·7126	–	–	38·276
Total nitrogen	0·0698	–	–	0·628

TABLE 1

200 gms Soil

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
			LIGHT		
0	1·4252	0·1396	...	Al., Va., Glu.	...
60	1·3527	0·1405	12·4	Al., Va., Glu.	...
120	1·3218	0·1408	11·6	Al., Va., Glu., Ly.	0·0309
180	1·2757	0·1412	10·7	Al., Va., Gly., Ly., Ar.	0·0348
240	1·2042	0·1417	9·5	Al., Va., Glu., Ly., Ar.	0·0334
			DARK		
0	1·4252	0·13960	...	Al., Va., Glu.	...
60	1·3765	0·14000	8·2	Al., Va., Glu.	...
120	1·3490	0·14021	8·0	Al., Va., Glu.	..
180	1·3186	0·14040	7·5	Al., Va., Glu. Ly.	0·0271
240	1·2864	0·14060	..	Al., Va., Glu., Ly.	0·0276

TABLE 2

200 gms Soil + Chlorella vulgaris

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
			LIGHT		
0	1·4252	0·1396	...	Al., Va., Glu.	...
60	1·3760	0·1406	20·3	Al., Va., Glu., Ly.	...
120	1·3512	0·1410	18·0	Pro., Al., Glu., Ly.	0·0366
180	1·3142	0·1414	16·2	Al., Va., Ly., Glu., Asp., Pro.	0·0361
240	1·2778	0·1419	15·6	Al., Va., Glu., Asp., Pro.	0·0347
			DARK		
0	1·4252	0·1396	...	Al., Va., Glu.	...
60	1·3763	0·1400	8·1	Al., Va., Glu.	...
120	1·3489	0·1402	7·8	Al., Va., Glu., Ly.	...
180	1·3183	0·1404	7·4	Al., Va., Glu., Ly.	0·0272
240	1·2963	0·1406	...	Al., Va., Glu., Ly.	0·0275

TABLE 3

200 gms Soil + Anabaena naviculoides

			LIGHT		
0	1·4252	0·1396	...	Al., Va., Glu.	...
60	1·3728	0·1411	28·6	Al., Va., Glu., Asp., Se.	...
120	1·3488	0·1416	26·1	Al., Va., Glu., Asp., Ar., Se.	0·0332
180	1·3114	0·1422	22·9	Al., Va., Glu., Asp., Ly., Ar.	0·0380
240	1·2771	0·1424	18·9	Al., Va., Glu., Asp., Ar.	0·0363
			DARK		
0	1·4252	0·13960	...	Al., Va., Glu.	...
60	1·3761	0·14002	8·5	Al., Va., Glu.	...
120	1·3490	0·14019	7·7	Al., Va., Glu.	...
180	1·3179	0·14042	7·6	Al., Va., Glu., Ly.	0·0272
240	1·2856	0·14063	7·3	Al., Va., Glu., Ly., Asp.	0·0275

TABLE 4

200 gms Soil + Tolypothrix tenuis

			LIGHT		
0	1·4252	0·1396	...	Al., Va., Glu.	...
60	1·3736	0·1412	31·0	Al., Va., Glu., Ly., Th., Asp.	...
120	1·3504	0·1417	28·0	Al., Va., Gly., Ly., Th., Asp.	0·0335
180	1·3126	0·1423	23·0	Al., Va., Glu., Ly., Th.	0·0382
240	1·2776	0·1425	19·6	Al., Va., Glu., Ly. Th., Asp.,	0·0364
			DARK		
0	1·4252	0·13960	...	Al., Va., Glu.	..
60	1·3755	0·14003	8·4	Al., Va., Glu.	...
120	1·3490	0·14019	7·7	Al., Va., Glu., Ly.	...
180	1·3180	0·14042	7·6	Al., Va., Glu., Ly.	0·0271
240	1·2854	0·14062	7·2	Al., Va., Glu., Asp	0·0275

TABLE 5

200 gms Soil + 1% C as Wheat Straw

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respcct to glycine
			LIGHT		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·0584	0·1833	28·3	Al., Va., Glu., Leu.	1·7946
120	2·8530	0·1887	27·6	Al., Va., Glu., Leu., Asp.	2·4632
180	2·6779	0·1935	27·5	Al., Va., Glu., Asp., Leu.,Ar.	2·7289
240	2·4709	0·1926	27·3	Al., Va., Glu., Asp.	2·6437
			DARK		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·1404	0·1774	15·8	Al., Glu., Va.	1·0721
120	2·9837	0·1797	15·4	Al., Glu., Va., Leu.	1·4897
180	2·8139	0·1821	15·3	Al., Glu., Va., Leu., Asp	1·9817
240	2·6415	0·1818	...	Al., Va., Glu., Leu., Asp., Ar.	1·8763

TABLE 6

200 gms. Soil + 1% C as Wheat Straw + Chlorella

			LIGHT		
0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·1412	0·1835	37·3	Al., Va., Glu., Leu., Pro.	1·8235
120	2·9852	0·1891	36·5	Al., Glu., Va., Leu., Pro.,Asp.	2·4867
180	2·8074	0·1939	33·9	Al., Va., Glu., Leu., Pro., Asp., Ar.	2·7469
240	2·6199	0·1923	32·7	Al., Va., Glu., Leu., Pro., Asp., Ar.	2·7231
			DARK		
0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·1405	0·1774	15·8	Al., Glu., Va., Leu.	1·0722
120	2·9836	0·1797	14·4	Al., Va., Glu., Leu., Al., Asp.	1·4896
180	2·8140	0·1821	15·3	Al., Va., Glu., Leu., Asp.,	1·9818
240	2·6414	0·1819	...	Al., Va., Glu., Leu., Asp.,Ar.	1 ·8763

TABLE 7

200 gms. Soil + 1% C as Wheat Straw + Anabaena

			LIGHT		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·1385	0·1860	45·6	Al., Glu., Va., Se., Leu.	2·0678
120	2·9341	0·1923	43·9	Al., Glu., Va., Se., Leu.	2·7423
180	2·8063	0·1998	43·4	Al. Va., Glu., Se., Leu., Asp., Ar.	2·9537
240	2·6188	0·1986	42·2	Al., Va., Glu., Se., Leu., Asp.	2·8234

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
DARK					
0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·1403	0·1774	15·8	Al., Glu., Va., Leu.	1·0722
120	2·9838	0·1797	15·4	Al., Glu., Va., Leu.	1·4895
180	2·8139	0·1821	15·3	Al., Glu., Va., Leu.	1·9820
240	2·6416	0·1819	...	Al., Glu., Va.	1·8759

TABLE 8

200 gms. Soil + 1% C as Wheat Straw + Tolypothrix

LIGHT					
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·1398	0·1867	48·3	Al., Glu., Va., Th.	2·3467
120	2·9845	0·1935	46·7	Al., Va., Glu., Asp., Th.,Leu.	2·9813
180	2·8069	0·2017	46·5	Al., Va., Glu., Asp., Leu.,Th.	3·1294
240	2·6196	0·2008	...	Al., Va., Glu., Th., Asp., Ly., Ar.	2·9672
DARK					
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·1404	0·1774	15·8	Al., Glu., Va., Leu.	1·0721
120	2·9836	0·1797	15·4	Al., Glu., Va., Leu.	1·4898
180	2·8140	0·1821	15·3	Al., Glu., Va., Leu.	1·9818
240	2·6416	0·1820	...	Al., Va., Glu.	1·8764

TABLE 9

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as Tata Basic Slag

LIGHT					
0	3·4252	0·1729	...	Al., Va.. Glu.	...
60	2·8628	0·1983	41·6	Al., Va., Glu., Leu.	2·5672
120	2·6685	0·2037	40·7	Al., Va., Glu., Leu., Asp.	3·7124
180	2·4005	0·2143	40·4	Al., Va., Glu., Leu., Asp., Ly., Hi., Ar.	4·4872
240	2·3013	0·2128	...	Al., Va., Glu., Leu., Asp., Ly., Hi., Ar.	4·2097
DARK					
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·0390	0·1830	26·1	Al., Glu., Va.	1·6698
120	2·8434	0·1877	25·4	Al., Va., Glu., Leu., Asp.	2·4367
180	2·6373	0·1911	23·0	Al., Glu.. Va., Leu., Asp.,Ar.	2·8189
240	2·4854	0·1907	...	Al., Glu., Va., Leu., Asp., Ar., Hi.	2·7496

TABLE 10

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as T. B. S. + Chlorella

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
			LIGHT		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	2·9684	0·1970	52·7	Al., Va., Glu., Pro.	2·5984
120	2·7160	0·2092	51·1	Al., Va., Glu., Pro., Leu.	3·7818
180	2·5374	0·2182	51·0	Al., Va., Glu., Pro., Leu., Hi., Ar., Asp., Ar.	45103
240	2·4173	0·2169	...	Al., Va., Glu., Pro., Leu., Hi., Ar., Asp.	4·2443
			DARK		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·3904	0·1830	26·1	Al., Va., Glu.	1·6689
120	2·84343	0·1877	25·4	Al., Glu., Va., Leu.	2·4369
180	2·63732	0·1911	23·0	Al., Va., Glu., Leu., Asp.,Hi.	2·8187
240	2·48547	0·1902	...	Al., Va., Glu., Leu., Asp., Ar. Hi.	2·7495

TABLE 11

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as T. B. S. + Anabaena

			LIGHT		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	2·9604	0·1985	55·0	Al. Va., Glu., Se.	2·8612
120	2·7152	0·2110	53·6	Al., Glu., Va., Se., Leu.	4·0391
180	2·5356	0·2191	51·9	Al., Va., Glu., Se., Leu., Hi., Asp., Ar.	4·8326
240	2·3866	0·2183	...	Al., Va., Glu., Se., Leu., Hi., Asp., Ar.	4·5349
			DARK		
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	3·0392	0·1830	26·1	Al., Va., Glu.	1·6696
120	2·84343	0·1877	25·4	Al., Va.. Glu., Leu.	2·4358
180	2·63738	0·1911	23·0	Al., Va., Glu., Leu., Asp.,Hi.	2·8189
240	2·48541	0·1907	...	Al., Va., Glu., Leu., Asp.. Hi., Ar.	2·7498

TABLE 12

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as T. B. S. + Tolypothrix

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
				LIGHT	
0	3·4252	0·1729	...	Al , Glu., Va.	...
60	2·9629	0·1992	56·8	Al., Glu., Va., Th.	3·1327
120	2·7159	0·2112	54·4	Al., Glu., Va., Th., Leu.	4·3672
180	2·5366	0·2200	53·0	Al., Glu., Va., Th., Leu., Asp., Ar., Hi.	5·1853
240	2·3580	0·2194	...	Al., Glu., Va., Th., Leu., Asp., Ar., Hi.	4·9401
				DARK	
0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·0392	0·1830	26·1	Al., Va., Glu., Leu.	1·6698
120	2·8434	0·1877	25·5	Al., Glu., Va., Leu.	2·4369
180	2·63736	0·1911	23·0	Al., Glu., Va., Leu., Hi.	2·8186
240	2·4854	0·1907	...	Al., Glu., Va., Leu., Hi , Ar.	2·7493

TABLE 13

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as German Basic Slag

				LIGHT	
0	3·4252	0·1729	...	Al., Va., Glu.	...
60	2·8340	0·1982	42·7	Al., Glu., Va., Leu.	2·7467
120	2·6559	0·2051	41·8	Al., Glu., Va., Leu., Asp.	3·8923
180	2·3974	0·2157	41·6	Al., Glu., Va., Leu., Asp., Ar., Hi.	4·5871
240	2·2784	0·2146	...	Al., Va., Glu., Leu., Asp., Ar., Hi.	4·4108
				DARK	
0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·0376	0·1831	26·2	Al., Va., Glu., Leu.	1·7224
120	2·8426	0·1877	25·4	Al., Va., Glu., Leu., Hi.	2·4178
180	2·6361	0·1913	23·3	Al. Va., Glu., Leu., Hi.	2·9119
240	2·4980	0·1904	...	Al., Va., Glu., Leu., Hi., Asp,	2·8274

TABLE 14

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as G. B. S. + Chlorella

				LIGHT	
0	3·4252	0·1729	..	Al., Va., Glu.	...
60	2·9600	0·1985	55·0	Al. Va., Glu., Pro.	2·7647
120	2·7189	0·2093	51·5	Al., Va., Glu., Pro., Leu.,Asp.	3·9353
180	2·5298	0·2187	51·1	Al., Va., Glu., Pro., Leu., Asp., Hi.	4·6158
240	2·3840	0·2179	...	Al., Glu., Va., Pro., Leu., Asp., Hi., Ar.	4·4549

Period of exposure in days	Total carbon gm.	Total nitrogen gm.	Efficiency	Aminoacids identified chromatographically	Amount of aminoacids with respect to glycine
				DARK	
0	3·4252	0·1729	...	Al., Glu., Va.	
60	3·03767	0·1831	26·2	Al., Va., Glu., Leu.	1·7226
120	2·8426	0·1877	25·4	Al., Va., Glu., Leu.	2·4181
180	2·6364	0·1913	23·3	Al., Glu., Va., Leu., Asp.,Hi.	2·9118
240	2·4986	0·1904	...	Al., Va., Glu., Leu., Asp.	2·8268

TABLE 15

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as G. B. S. + Anabaena

LIGHT

0	3·4252	0·1729	...	Al., Glu., Va.	...
60	2·9581	0·1996	57·1	Al., Glu., Va., Se.	3·2659
120	2·7163	0·2117	54·6	Al., Glu., Va., Se., Leu.,Asp.	4·4165
180	2·5280	0·2204	52·9	Al., Va., Glu., Se., Leu., Asp., Hi.	4·8724
240	2·3828	0·2197	...	Al., Va., Glu., Se., Leu., Asp., Hi.	4·6 89

DARK

0	3·4252	0·1729	...	Al., Glu., Va.	..
60	3·03762	0·1831	26·2	Al., Glu., Va., Leu.	1·7226
120	2·84262	0·1877	25·4	Al., Glu., Va., Leu.	2·4181
180	2·6364	0·1913	23·3	Al., Glu., Va., Leu., Asp.	2·9119
240	2·4986	0·1904	...	Al., Glu., Va., Leu., Asp., Hi.	2·8273

TABLE 16

200 gms. Soil + 1% C as Wheat Straw + 0·5% P_2O_5 as G. B. S. + Tolypothrix

LIGHT

0	3·4252	0·1729	...	Al., Glu., Va.	...
60	2·9584	0·2004	58·8	Al., Glu., Va., Leu., Th.	3·3729
120	2·7180	0·2135	57·4	Al., Glu., Va., Leu., Th., Asp., Ar.	4·5237
180	2·5289	0·2210	53·6	Al., Va., Glu., Leu., Th., Asp., Ar., Hi.	4·9635
240	2·3837	0·2205	...	Al., Glu., Va., Leu., Th., Asp., Ar., Hi.	4·6812

DARK

0	3·4252	0·1729	...	Al., Glu., Va.	...
60	3·03761	0·1831	26·2	Al., Glu., Va., Leu.	1·7225
120	2·84260	0·1877	25·4	Al., Glu., Va., Leu., Asp.	2·4179
180	2·63612	0·1913	23·3	Al., Glu., Va., Leu., Asp.	2·9121
240	2·49801	0·1904	...	Al., Glu., Va., Leu., Asp., Hi.	2·8272

Discussion

It is observed that there is an appreciable oxidation of carbon from wheat straw, when mixed with soil and allowed to undergo slow oxidation in air at the laboratory temperature, more in the light than in the dark sets. There is concomitant increase in the number and amount of aminoacids formed along with nitrogen fixation, which is also greater in the sets exposed to light than in the dark. These observations show that light energy is helpful in nitrogen fixation and subsequent utilization of the fixed nitrogen in the synthesis of aminoacids.

It is very interesting to note that the efficiency of nitrogen fixation *i.e.* the amount of nitrogen fixed in milligram per gram of carbon oxidised is always greater in the sets exposed to light than those kept in the dark and the efficiency gradully decreases, the reasons are clear from the following consideration. In our previous paper[13] it is shown, that in soils, the process of nitrogen fixation is always opposed by ammonification and nitrification. Thus on one hand, nitrogen fixed on the surface of the soil undergoes various changes aided by light absorption to form aminoacids and proteins and on the other hand these aminoacids and proteins formed in the soil or sand undergo ammonification and nitrification which are accelerated by light absorption and form nitrite and nitrate in the following manner :—

$$\text{Proteins} \longrightarrow \text{Aminoacids} \xrightarrow{+O_2} NH_4 \text{ compounds} \xrightarrow{+O_2} NO_2 \xrightarrow{+O_2} NO_3$$

In these series of reactions the unstable ammonium nitrite is formed which undergoes decomposition liberating energy and nitrogen gas as in the following equation :—

$$NH_4NO_2 \rightarrow N_2 + 2H_2O + 718 \text{ K. cal.}$$

But the addition of phosphate *i.e.* Tata or German basic slag to the systems, form stable phosphoproteins or phospho ligno proteins which resist the ammonification and nitrification processes. It is clear from the results that the amount and number of aminoacids is greater in phosphated than in the unphosphated systems. It is interesting to record that in the sets inoculated with algae some saving in carbon and a small increase in nitrogen was always found in light though in the dark, there was no significant difference in the carbon and nitrogen status of the systems. The order of carbon saving was Chlorella>Tolypothrix>Anabaena. In sets inoculated with chlorella, there is a slight increase in nitrogen compared with those which have not been inoculated. It is interesting to note that though chlorella is not a fixer of nitrogen, even then it shows a small fixation of nitrogen in light. The probable reason of this seems to be that in soils chlorella lives in symbiosis with nitrogen fixing organism azotobacter[14] and during this process it supplies the azotobacter with carbohydrates and the bacteria in turn fixes nitrogen.

In the sets inoculated with tolypothrix and Anabaena, there is appreciable increse in the nitrogen content over that of control sets as well as in sets inoculated with chlorella, thereby having better efficiencey, the increase being greater in the case of tolypothrix than in that of anabaena. This is in agreement with the general observation made by various algologists that tolypothrix is a better fixer of nitrogen than anabaena.

The systems inoculated with algae contain greater amount and number of aminoacids than is contained in the uninoculated systems. This is probably due to decomposition and oxidation of algal bodies which increases the amount and number of aminoacids. It has been generally observed that chlorella favours the formation of proline, while in systems having anabaena and tolypothrix serine and threonine have generally been detected. The amount of aminoacids is always larger in sets inoculated with anabaena and tolypothrix than in the sets containing chlorella. The possible reason for this seems to be that anabaena and tolypothrix fix atmospheric nitrogen along with carbon saving, the fixed nitrogen is utilized in the synthesis of aminoacids and some of the aminoacids are excreted in the medium. Fowden[7], Watanabe[6] and Williams and Burris,[8] found many free aminoacid present in algae. Rothwell and Frederick[4] observed aminoacids during the decomposition of alfalfa and corn stover in soil at different temperatures.

References

1. Leibniz. *Historia Inventionis Phosphorie, Berlin,* 1710.
2. Krauss, R. W. *Ann. Rev. Plant Physio,* **9**, 207–244, 1958.
3. Stevenson, F. J. *Proc. Soil Sci. Soc. Amer.*, **20**, 204, 1956.
4. Rothwell, D. F. and Frederick, L. R. *Proc. Soil Sci. Soc.*, **16**, 157, 1956.
5. Bremner, J. M. *J. Sci. Fd. Agri.*, **3**, 497, 1952 ; *J. Soil Sc.*, **2**, 67, 1951 and **5**, 214, 1954.
6. Watanabe, A. *Arch. Biochem.*, **34**, 50–55, 1951.
7. Fowden, L. *Nature*, **167**, 1030-31, 1951.
8. Williams, A. E. and Burris, R. H. *Amer. J. Bot.*, **39**, 340–42, 1952.
9. Dhar, N. R. and Roy, T. B. G. *Proc. Nat. Acad. Sci. India,* **24A**(3), 1955.
10. Simonart, P. and Peeters, F. *Trans. 5th Intern. Congr. Soil, Sci.*, **3**, 132–135, 1954.
11. Payne, T. M. B. and Rouatt, J. W. and Katznelson, H. *Soil Sci.*, **82**, 521-524, 1956.
12. Harding and Mclean. *J. Biol. Chem. Soc.*, **20**, 217, 1915 ; **24**, 503, 1916.
13. Dhar, N. R. and Arora, S. K. *Proc. Nat. Acad. Sci., India, (in press)*
14. Russell, E. J. Soil conditions and plant Growth, 335–36, *ed.*, 1950.

Proc. Nat. Acad. Sci., India, 40(A), IV, 1970

Oxidation of α-Hydroxy acids by Vanadium (V)

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Abstract

The stoichiometry of the oxidation of α-hydroxy acids by vanadium (V) under different conditions of mineral acid concentrations has been established potentiometrically using platinum and mercury coated platinum electrodes. At low H_2SO_4 concentrations 2 equivalents and at high H_2SO_4 concentrations 4 equivalents of the oxidant are used up per mole of the hydroxy acid, yielding corresponding aldehydes and carboxylic acids respectively. At intermediate concentrations of H_2SO_4 the lack of stoichiometry is due to two simultaneous competitive reactions leading to mixed products. These results are explained on the basis of the standard oxidation potential data. Products in some cases have been isolated and identified.

The reactions have analytical significance.

In the oxidation of lactic acid by periodic acid Brown[1] realised all the successive oxidation stages (pyruvic acid, acetic acid, formic acid and finally CO_2). In the kinetic studies of the oxidations of α-hydroxy acids by different oxidising agents, other workers[2-6] have assumed the oxidation products to be α-keto acids, aldehydes or simple carboxylic acids depending upon the nature of the oxidant and the concentration of the mineral acid used.

In the present paper, the stoichiometry of the oxidation of lactic and mandelic acids with vanadium (V) under varying concentrations of mineral acids has been determined potentiometrically in addition to exploring analytical prospects of such oxidations.

Experimental

M/100 solutions of $NaVO_3$ (Analar) were prepared, in different concentrations of H_2SO_4 and standardised with Mohr's salt solution using N-phenyl anthranilic acid as indicator. M/20 solutions of lactic and mandelic acids were prepared in freshly distilled air free water and standardised with carbonate free NaOH solutions.

For potentiometric investigations, 20 ml. of the acid vanadate + x ml. of α-hydroxy acid, x varying from 0 to 2, were taken in a series of bottles. After vigorous shaking, the raction mixtures were allowed to remain at 28°C for 24 hours to attain equilibrium. The indicator electrode, in the case of mandelic acid, was a bright platinum foil and in the case of lactic acid was a mercury coated thick platinum wire. While in the case of lactic acid the bright platinum foil failed to detect any point of inflexion, in the case of mandelic acid both the electrodes worked satisfactorily, the mercury electrode proving better (vide fig. 3). E. m. f's were determined by immersing the electrode in each bottle, one by one, and combining it with a saturated calomel electrode.

The Hg-coated Pt electrode was prepared by the method described by Strouts, Gilfillen and Wilson.[7]

The Pt electrode, prior to its use, was cleaned in chromic acid followed by treatment with (N) NaOH and subsequent washing with distilled water and glow-in alcohol flame.

FIG.-1

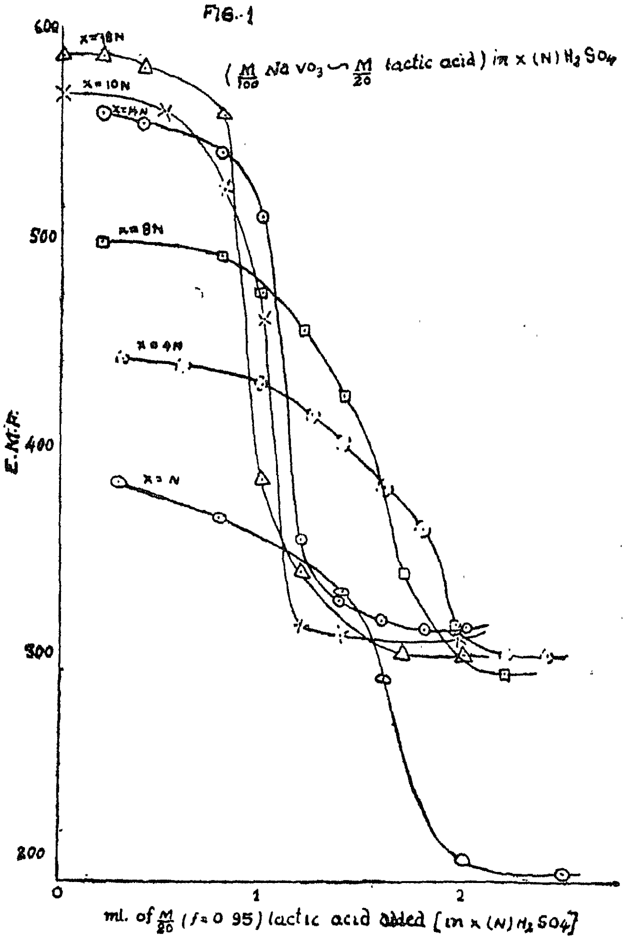

Fig-2

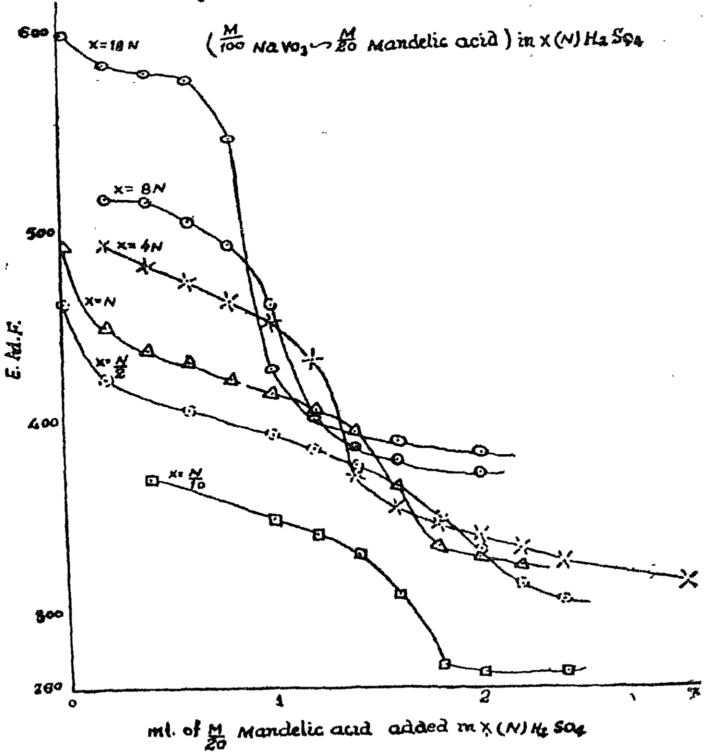
$(\frac{M}{100}\ NaVO_3 \backsim \frac{M}{20}$ Mandelic acid) in x(N) H_2SO_4

Fig-3

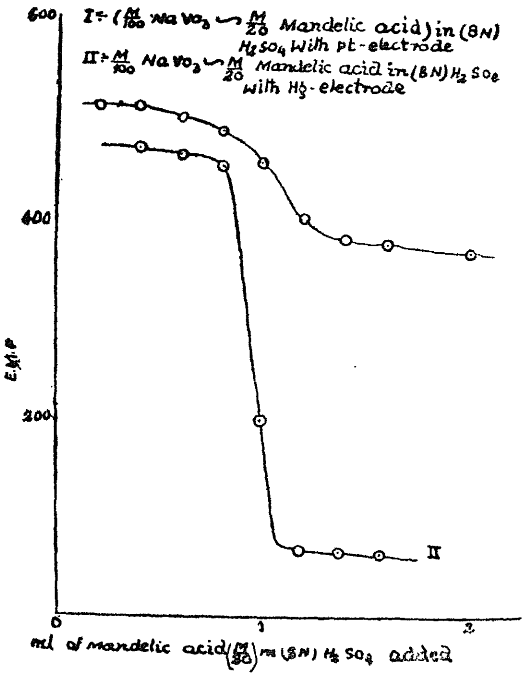

TABLE 1

20 ml. of x(M/100) $NaVO_3$ in y(N) $H_2SO_4 \equiv$ z ml. of M/20 α-hydroxy acid.
Temperature—28°C

α-hydroxy acid	x	y	z	moles of $NaVO_3$ per mole of the α-hydroxy acid
Lactic acid 0·95 (M/20)	0·901	0·5	1·9	2·00
	0·901	1	1·9	2·00
	0·901	4	1·9	2·00
	0·950	8	1·7	2·35
	0·950	10	1·2	3·33
	0·950	14	1·1	3·63
	0·960	18	1·0	4·00
Mandelic acid (M/20)	0·98	0·1	1·8	2·1
	0·98	0·5	1·8	2·1
	0·98	1	1·8	2·1
	0·98	4	1·4	2·7
	0·98	8	1·1	3·45
	0·98	18	1·0	3·92

Volumetric determination of the α-hydroxy acids by oxidations with acid vanadate :

10 ml. of a solution containing a known weight of the α-hydroxy acid and 30 ml. of 0·95 (M/10) $NaVO_3$ were mixed to which sufficient amount of concentrated H_2SO_4 was added so as to bring the acid concentration above 14N. After thoroughly shaking and leaving the reaction mixture for 24 hours, the solutions were diluted to 100 ml. The excess of vanadate in solutions was back titrated with a standard Mohr's salt solution using N phenyl anthranilic acid as indicator. At the end point the colour change was from violet to green.

TABLE 2

H_2SO_4 concn.	α-hydroxy acid taken	Found	Error
16 N	45 mg. of lactic acid	44·1 mg.	2%
18 N	45 mg. of lactic acid	44·4 mg.	1·33%
16 N	76 mg. of mandelic acid	74·95 mg.	1·38%
18 N	76 mg. of mandelic acid	75·1 mg.	1·18%

Results and Discussions

Preliminary investigations as well as results in Table 1 indicate that both the rate and stoichiometries of the present oxidations are mineral acid dependent, the oxidation of the mandelic acid being faster than that of the lactic acid. In the complete absence of mineral acids, no reaction occurs ; at concentrations below N/2, no inflexion or vague ones were located in the e. m. f. curves. At

H_2SO_4 concentrations N/2 to 4N, 2·0 moles, and above 14N, 4·0 moles of vanadium (V) are consumed per mole of the hydroxy acid. At intermediate concentrations of H_2SO_4 the moles of vanadium (V) consumed per mole of the hydroxy acid are not integral, though the value increases and approaches a limiting one of 4·0, as the H_2SO_4 concentration increases. Assuming one equivalent reduction of vanadium (V) in each case, these results correspond to the following overall redox processes,

$$\text{At high } H_2SO_4 \text{ concn., } RCHOHCOOH + 4VO_2^+ + 4H^+ = RCOOH + CO_2 + 3H_2O + 4VO^{++} \qquad (i)$$

$$\text{At low } H_2SO_4 \text{ concn., } RCHOHCOOH + 2VO_2^+ + 2H^+ = RCOCOOH + 2H_2O + 2VO^{++}$$

$$\text{or} \quad RCHOHCOOH + 2VO_2^+ + 2H^+ = RCHO + CO_2 + 2H_2O + 2VO^{++} \qquad (ii)$$

No significant change was noticed, except in the values of e. m. f., by substituting HCl or $HClO_4$ for H_2SO_4. The use of H_3PO_4 was difficult because the vanadate solution tended to through a yellow precipitate, presumably a heteropoly acid.

At low acid-concentrations, the reaction mixture, in case of mandelic acid, soon smelt of benzaldehyde which was isolated in the end and characterised as its 2 : 4-dinitrophenyl hydrazone (m. p. 235°C). In the case of lactic acid the acetaldehyde formed was isolated and characterised as its dimedone derivative (m. p. 140°C). In the case of mandelic acid, at high H_2SO_4 concentrations the product, benzoic acid, was isolated, recrystallised and its m. p. checked. Any α-keto acid formed as an intermediate is, therefore, ultimately, with lapse of time, converted to the aldehyde without affecting the stoichiometry of the redox process.

At intermediate acid concentrations (4N to 14N) the lack of stoichiometry is presumed to be due to two simultaneous competitive reactions (*i*) and (*ii*) leading to mixed products. The fact that in no case more than one inflexion is noticed in the e. m. f. curves leads to the conclusion that the same equilibrium constant covers both products.

Oxidation to formic acid (in case of lactic acid) and CO_2, requiring 10 and 12 equivalents of the oxidant respectively, is not indicated even at acid concentrations greater than 14N.

The acid dependence of the reaction rate is understood in terms of the existence of an equilibrium

$$VO_2^+ + H_3O^+ = V(OH)_3^{++}$$

followed by the attack of the doubly charged cation which is known to be a more powerful oxidiser than the VO_2^+ itself[8]. The range of redox process which can then occur is limited by thermodynamic considerations. On the basis of the potentials of the lactic acid—pyruvic acid couple[9] ($E^\circ = -0{\cdot}20$ volt) and acetaldehyde—acetic acid couple[10] ($E^\circ = 0{\cdot}118$ volt) it should be relatively easy to oxidise lactic acid to acetic acid than to pyruvic acid (or acetaldehyde). Yet both vanadate and dichromate[5], which differ in their E° values by 0·30 volt only, oxidise, at low acid concentrations, lactic acid to pyruvic acid (or acetaldehyde), oxidation to acetic acid occurring only at high H_2SO_4 concentrations. With respect to the reduction of vanadate, larger decreases of free energy are expected in the reduction to V(IV) (E° = -1·000 volt) rather than to V(III) (E° = -0·68 volt). Moreover, comparing with other weak 2-equivalent oxidants like I_2 (E° = -0·53 volt) or arsenic acid (E° = -0·559 volt), the V(III) -V(IV) couple is not likely to be involved in organic oxidations.

The stoichiometric oxidations of the α-hydroxy acids at high H_2SO_4 concentrations raise the prospects of their analytical applications. An attempt has been made to evolve a method with an error of less than 2% (Table 2).

References

1. Brown, *Chem. Abs.*, **31**, 4965, 1937.
2. West and Skoog, *Anal. Chem.*, **31**, 583, 1959.
3. Jones, Litler and Waters, *J. Chem. Soc.*, 630, 1961.
4. Rama Shankar and S. N. Swami, *J. Indian Chem. Soc.*, **40**, 105, 1963.
5. Gyani, B. P. and Prasad, S. N., *J. Indian Chem. Soc.*, **39**, 965, 1962 ; **41**, 156, 1964.
6. Krishna and Tewari, *J. Chem. Soc.*, 3097, 1961.
7. Strout, Gilfillen and Wilson. Chemical Analysis, Vol. II, Clarendon Press, Oxford, 1962, p. 41.
8. Ramsay, Colichman and Pack, *J. Amer. Chem. Soc.*, **68**, 1695, 1946.
9. Wurmser and Mayer-Reich, *Compt. Rend.*, **196**, 612, 1933.
10. Latimer, Oxidation Potentials, Prentice-Hall, Inc. N. J. p. 131, 1959.

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Effect of Direct and Indirect application of Organic Matter (phosphated and unphosphated) on the yield of wheat crop

By

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The soil fertility is impoverised by continuous cropping. The capacity of the soil to supply nutrient elements to plants in required amounts and balanced proportions is commonly termed as "Soil fertility". It is governed by several factors. Of them, most important is the organic matter. As soil organic matter, in addition to amelioration of physical properties of soil, improves nutrient status of the soil, when it undergoes slow oxidation in the soil. A majority of soil scientists, have emphasised a need of replenishing the supply of organic matter in the soil. Karunakar, R. D. (1951) pointed out that organic matter adds nutrients to the soil. It has also been revealed by a number of soil scientists that organic matter chelates the nutrients available in the soil and thus checks the loss of available nutrients from soil.

In Indian soils organic matter is lower as compared to the soils of European countries. It ranges from 0·4 to 0·6%. In view of the vital role of the organic matter in the land fertility and its general deficiency in our soils, one of the major problems of Indian Agriculture is to build up soil organic matter and to conserve it.

The sources of the organic matter are crop residues, green manures, animal refuse, weeds and tree leaves. Out of these, crop residues can not be utilised by transporting to another field, and green manuring is not always profitable due to usual corp rotations. The animal refuse, weeds, tree leaves have other uses like fuels than as manure. Composting is advocated for this purpose, yet composts have a doubtful utility in the building up the organic matter content of soil. It is because of heavy losses of nitrogen during composting process which is reported by many scientists mainly Albrecht and Poirst, Gilbert, Pember, Gerlach etc.

Dhar and co-workers have found that direct application of plant materials to the fields, instead of composting, is more beneficial to crops because the energy materials like carbohydrates, fats, lignin etc. when added to the soil, are slowly, oxidised. The energy liberated helps in fixing atmospheric nitrogen on the soil surface, forming NH_3, Amino acids and other nitrogenous compounds.

A study of comparing organic matter used directly and after composting has been made by us in pots in the green house of the Institute under controlled conditions in the wheat.

Experimental

Soil of top 9" was taken, dried, powdered, cleaned from foreign materials mixed well before use. 6 kgs of this soil were taken in which organic marerials *e. g.* Paddy straw, water hyacinth and weed (after thorough mixing, powdering, and sieving through 100 mesh) were added @ 10 tons/acre. Phosphate was added as G. B. S. @ 50 lbs. P_2O_5 percent. For indirect method composts (phosphated with G. B. S. containing 17·4 % P_2O_5) were already prepared. The rate of compost application was on nitrogen basis which was 100 lbs. per acre.

After thorough mixing the mixture was filled in pots and the system was kept moist with 20% moisture level by weight. In case of direct application the pots were exposed to sunlight for one month with mixing of materials at an interval of a week, where as in composting the soil and composts were thoroughly mixed and exposed for 15 days with two mixings and 20% moisture level. Then the wheat crop was sown in both the systems on the same date.

Scheme :

1. Soil alone
2. Soil + mixed leaves (weeds)
3. Soil + weed + G. B. S. @ 1% P_2O_5.
4. Soil + weed + G. B. S. @ 0·5% P_2O_5.
5. Soil + water hyacinth
6. Soil + water hyacinth + G. B. S. @ 1% P_2O_5.
7. Soil + ,, ,, + ,, @ 0·5% P_2O_5.
8. Soil + Paddy Straw
9. Soil + ,, ,, + ,, @ 1% P_2O_5.
10. Soil + ,, ,, + ,, @ 0·5% P_2O_5.

Treatments were the same in direct and indirect applications.

Replications	4
Total no. of pots	40
Design	Randomised Block Design
Date of sowing	15th Oct. 1968
Date of harvesting	28th April 1969
Variety	K–68
Plant population (per pot)	5

Materials taken for the application :

Organic matters :

Analysis	Water hyacinth	Paddy straw	Weed (Mixed)
Loss on ignition	63·241	86·23	89·45
Ash	38·53	13·04	10·13
Fe_2O_3	1·34	1·33	0·42
Al_2O_3	0·089	–	–
CaO	2·263	0·66	0·74
MgO	0·785	0·169	0·708
K_2O	4·684	1·706	0·983
P_2O_5	0·563	0·710	0·526
Total carbon	32·44	31·996	41·349
Total nitrogen	1·514	0·580	0·659
C/N ratio	21·56	55·17	63·61

Analysis of German Basic Slag which is used as phosphate

Silica	13·227
Fe_2O_3	16·136
Total P_2O_5	17·853
Available P_2O_5	8·997
Total CaO	41·132
Total K_2O	0·184
Total MgO	5·103

Analysis of compost prepared (phosphated and unphosphated)

	Phosphated with G. B. S.		
Analysis	1% level	0·5% level	Unphosphated
	Mixed weeds compost		
Total carbon	12·93	16·04	25·01
Total nitrogen	1·68	1·49	1·16
NH_3-N	0·062	0·043	0·0431
NO_3-N	0·1214	0·102	0·0896
Available P_2O_5	0·721	0·385	0·282
	Paddy straw compost		
Total carbon	9·75	11·03	19·68
Total nitrogen	1·58	1·36	0·93
NH_3-N	0·084	0·046	0·271
NO_3-N	0·139	0·963	0·072
Available P_2O_5	0·649	0·341	0·2639
	Water hyacinth compost		
Total carbon	16·491	17·680	20·143
Total nitrogen	2·594	2·406	1·937
NH_3-N	0·136	0·1203	0·0814
NO_3-N	0·198	0·2013	0·1030
Available P_2O_5	0·697	0·335	0·6910

Average Yield of wheat crop (in grams)

Treatments	Direct (grain)	Indirect (grain)
Soil alone	53·1	50·6
Soil + mixed weed	136·3	68·5
Soil + mixed weed + 1% P_2O_5	151·3	79·3
Soil + mixed weed + 0·5% P_2O_5	146·2	72·2
Soil + water hyacinth	149·6	73·8
Soil + water hyacinth + 1% P_2O_5	167·4	89·5
Soil + water hyacinth + 0·5% P_2O_5	152·9	76·8
Soil + Paddy straw	131·4	62·0
Soil + Paddy straw + 1% P_2O_5	146·2	70·1
Soil + Paddy straw + 0·5% P_2O_5	139·5	67·5

Discussion

An examination of the yield data manifests a marked increase in yield of wheat crop in all treatments in which organic materials such as paddy straw, mixed weeds, water hyacinth have been added either directly or indirectly over control. Moreover, when organic matter is applied in conjunction with calcium phosphates German Basic Slag higher yields were obtained over the control as well as even the treatments in which only organic matter was added.

Dhar and his coworkers have established this that slow oxidation of organic matter fixes atmospheric nitrogen. Further they have also proved that addition of phosphates in conjunction with organic matter stabilises the fixed atmospheric nitrogen due to phosphorylation. From this fact, it can be stated that increase of wheat yield, in treatments in which organic matter and organic matter with phosphates were added, is due to fixation of atmospheric nitrogen as well as the availability of phosphate, calcium, Mg., potassium and to some extent trace elements due to dissolution of basic slag in the soil on account of liberating of organic acids chiefly carbonic acid owing to slow oxidation of organic matter in soil.

Direct application of organic matter (phosphated and unphosphated) is better than that of indirect (compost) because in case of direct application more nitrogen was fixed in the system. The whole of fixed nitrogen was utilized by plants grown. But, during composting, the fixed nitrogen was lost in the form of gases and leaching (reported by Dhar and Coworkers).

With forgoing discussions it can be concluded that direct application of organic matter both phosphated and unphosphated is superior to indirect application and at the same time higher doses of phosphorus was found to be superior to lower doses.

References

1. Albrecht, W. A. *U. S. D. A. year book*, 347-360, 1938.
2. Agrawal, R. R. and Singh, M. and Chandra, P. *U. P. Agric. Deptt. Bull.*, No. 104, 1950.
3. Dhar, N. R. *Presi. addr. Ind. Sci. Cong. 48th, Session*, p. 11, 1966.
4. Dhar, N. R. and Sharma, D. N. *D.Phil. Thesis, University of Allahabad*, 1965.
5. Ghose, S. K. *J. Ind. Soc. Soil Sci.*, **11**(2), 129-135, 1963.
6. Ray Chaudhuri, S. P. *Proc. Symp. Nat. Inst. Sci., N. Delhi* 1962. *Bull., No.* 26, 1964.

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Study of Manganese Status of some deficient Medium Black Soils of Madhya Pradesh

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Abstract

Surface soil samples (0–9″) from 153 villages of four sub-divisions of Indore district *viz.*, Mhow, Depalpur, Indore and Sawer which is a Manganese deficient area of the State, were analysed for exchangeable and reducible Manganese and for Cal. Carbonate, Organic Carbon and pH. The soils are found to be deficient in available Manganese but high in reducible Manganese. The soils of some of the villages in the locality were found to be acutely deficient in available Manganese.

Correlation studies indicated that available Manganese bears a negative relationship with pH and Cal. Carbonate but a positive relationship with Organic Carbon and reducible Manganese content of the soil. The correlation between available and reducible Manganese was found to be highly significant.

The importance of Micro-nutrients in Indian Agriculture can not be over emphasised especially in view of the efforts being made to improve Agriculture on scientific lines and on intensive cropping with high doses of fertilizers for increasing production per unit area per unit time. In recent years, the importance of the Micro-nutrients in increasing Agricultural production and correcting deficiencies in plants has been recognised by soil scientists all over the world. In the list of essential micronutrients, Manganese occupies a place second only to iron in order of importance for plant growth. Its availability directly depends on the soil pH, $CaCO_3$ and Organic Carbon content (Leeper, 1947. Biswas, 1951, Khanna, 1954). The medium black soils of India are rich in total Manganese but low in available Manganese (Biswas 1951, 1953). It was also observed by earlier workers that the medium black cotton soil of Malwa plateau is deficient in available Manganese (Sharma and Motiramani 1964). The present study was conducted to find out the quantity of the two different forms of Manganese (Water Soluble + Exchangeable *i. e.* available and reducible Mn) in some medium black soils of Madhya Pradesh and to determine the relationship between different soil properties and reducible Manganese with available Manganese.

Experimental

The surface soil samples were collected from the representative soils of 153 villages of Indore District. The samples were analysed for available Manganese (IN. Ammonium Acetate pH 7·0), reducible Manganese (IN. NH_4 Acetate pH 7·0 + 0·2% hydroquinone), pH (1 : 2·5 soil water ratio by Beckman pH meter using glass electrode) as per procedures described by Jackson (1958) Organic Carbon (Walkley and Black's procedure, 1934) and Calcium Carbonate (rapid titration procedure as described by C. S. Piper, 1950). Estimation of Manganese was done by the Colorimetric procedure (Willard and Greathouse, 1917).

Results and Discussion

The Manganese deficient medium black soils of M. P. which were taken up for study are clayey in texture with high calcium carbonate content. They are low in Organic Carbon content as shown in Table 1 (Muhr *et al.* 1965).

TABLE 1

Soil Characteristics (Average Values)

S. N.	Sub-Division	No. of Samp.	pH	$CaCO_3$%	O. C.%
1.	Mhow	60	8·0	25·0	0·460
2.	Depalpur	55	8·1	23·0	0·440
3.	Indore	53	8·1	23·0	0·455
4.	Sawer	41	8·2	22·0	0·445
Average of Total		209	8·1	22·0	0·450

Distribution of Manganese

Manganese status of soils is presented in Table 2. The available Manganese of these soils varied from 1·92 to 9·62 ppm with an average of 5·05 ppm for the district. Thus it can be safely said that these soils are poor in available Manganese (Toth-1951, Sharma and Motiramani, 1964).

TABLE 2

Distribution of Manganese (Available and Reducible)

S. N.	Sub-Division	No. of Samps.	Available Manganese Range (ppm)	Available Manganese Aver (ppm)	Reducible Manganese Range (ppm)	Reducible Manganese Aver (ppm)
1.	Mhow	60	1·92–9·62	6·01	21·16–481·0	255
2.	Depalpur	55	2·16–5·20	5·2	48·10–481·0	258
3.	Indore	53	2·16–8·42	4·7	48·10–408·0	202
4.	Sawer	41	1·92–8·42	4·4	120·0 –425·0	198

The easily reducible Manganese content in the entire district ranges between 21·16 and 481 ppm with an average value of 236·43 ppm. Most of the samples showed a higher amount of easily reducible Manganese content (Leeper, 1947).

TABLE 3

Distribution of available Manganese Sub-division-wise

Range of Mn (ppm)	Mhow No. of Samp	Mhow %	Depalpur No. of Samp	Depalpur %	Indore No. of Samp	Indore %	Sawer No. of Samp	Sawer %
1·92–3·84	17	28%	7	12%	15	28%	16	39%
3·84–6·40	17	28%	35	63%	33	64%	23	56%
6·40–9·62	26	44%	13	25%	5	8%	2	5%

Table 3 gives a comprehensive view of the distribution of samples under different ranges of percentages. Study of Table 3 reveals that all the samples have less than 10 ppm of available Manganese content showing thereby, that the entire area in Indore District is deficient in available Manganese content. It will be seen that more than 50% samples of each sub-division have less than 6·4% Available Manganese. However, the soils of Sawer, Indore and Depalpur, sub-divisions are comparatively more deficient than of Mhow sub-division. Sharma and Shinde (1968) have also reported similar results in black soils of Indore region. The results reported here also indicate that these soils are poor in available Manganese but high in reducible Manganese.

Relationship of Available Manganese with reducible Manganese and other soil components

The relationship of Available Manganese with reducible Mangenese and other soils components such as pH, $CaCO_3$ content and Organic Carbon content is given below in Table 4 after working out the correlation coefficients.

TABLE 4

Correlation between Available Manganese and different soil components

S. No.	Correlation Between	r-Values
1.	Available Manganese and soil pH	-0·12
2.	Available Manganese and $CaCO_3$ content.	-0·129
3.	Available Manganese and Organic Carbon content.	+0·135
4.	Available Manganese and Reducible Manganese.	+0·382*

*Significant at 1% level.

The correlation studies made between Available Manganese and other soil characteristics show that there is a negative correlation with pH and $CaCO_3$ but a positive correlation with Organic Carbon and Reducible Manganese. However, the coefficient of correlation was found to be statistically significant, only in the case of Reducible Manganese.

The negative correlation in case of pH is in conformity with the observations of Berger (1948) Jones (1957), Zende *et al.* (1959) Randhawa *et al.* (1961), Mittal and Roy (1963) and also of previous workers in this laboratory *viz*; Sharma and Motiramani (1964) Dixit (1965), and Sharma (1966) who also found it to be highly significant statistically. This fact could be explained on the basis that at low pH, Manganese is present as di-valent Manganous ion which has higher solubility (Hewitt-1946), Berger and Gerloff-1948 and Biswas-1953) but with the increase in pH it gets converted into trivalent and tetravalent ions of Manganese which get precipitated as insoluble oxides and thus get converted into unexchangeable and therefore unavailable form (Sherman and Harmer-1942, Biswas-1953). Low values of exchangeable Manganese have also been reported in saline soils of Rajasthan, Punjab and U. P. by Vinayak *et al.* (1964), Bhumbla and Dhingra (1964) and Agarwala *et al.* (1964) respectively.

The negative correlation observed in the case of $CaCO_3$ percentage is also a harmony with the results reported by Yadav and Kalra (1964) Sherman and Fujimoto (1947) and Lohnis (1960) who reported decrease in available Manganese on addition of lime to Acid soils and also by previous workers in this laboratory *viz* ; Sharma and Motiramani (1964), Dixit (1964) and Sharma (1966) who also observed a negative significant correlation between available Manganese and $CaCO_3$ content of soils. It is also in keeping with the negative relationship of Available Manganese with pH as pH is directly related with $CaCO_3$ content of soil.

The positive correlation observed in the case of Organic matter is not statistically significant. This is in conformity with the results of some earlier workers in this laboratory *viz* ; Sharma *et al.* (1964) and Dixit (1965) but is contradictory to the observations of Sharma (1966). In the literature also contradictory results have been reported by different workers (Kanwar and Randhawa 1967). Hence the role of Organic Matter in the conversion of Manganese oxides is not clear and there seems to be no significant relationship between Available Manganese and Organic Matter, especially when Organic Matter content is poor as is the case here. The positive correlation between Available and Reducible Manganese is also found to be highly significant statistically. This observation is also in agreement with the observations of earlier workers *viz.*, Sharma *et al.* (1964), Dixit (1965) and Sharma (1966). Thus, there seems to exist a direct relationship between these two forms of Manganese, both of which are beneficial for plant growth (Jones and Leeper–1951).

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References

1. Agarwala, S. C. Mehrotra, N. K. and Sinha, B. K. *J. Indian Soc. Soil Sci.*, **12**, 7, 1964.
2. Berger, K. C. and Gerloff, G. C. *Soil Sci. Soc. Amer. Proc.*, **12**, 310, 1948.
3. Berger, K. C. *Amer. Potato, J.*, **25**, 47, 1948.
4. Biswas, T. D. *Indian J. Agr. Sci.*, **21**, 97, 1951.
5. Biswas, T. D. *J. Indian Soc. Soil Sci.*, **1**, 21, 1953.
6. Bhumbla, D. R. and Dhingra, D. R. *Ibid*, **12**, 255, 1964.
7. Dixit, R. G. *M.Sc. (AG) Thesis of the J. N. Agri. Univ. Jabalpur*, 1965.
8. Hewitt, E. J. *Annual Rept. Agri. Hort. Res. Stn. Long Ashton, Bristol (England)* 50-61, 1946.
9. Jones, L. P. M. and Leeper, G. W. *Plant and Soil*, **3**, 154, 1951.
10. Jackson, M. L. Soil Chemical Analysis, (*Prentice Hall, Inc., N. Y.*) 1958.
11. Khanna, K. L., Bhattacharya, P. B. and Joneja, K. L. *J. Indian Soc. Soil Sci*, **2**, 105, 1954.
12. Kanwar, J. S. and Randhawa, N. S. *Micronutrient Res. in Soil and Plants in India—A review (ICAR) New Delhi*, 1967.
13. Leeper, G. W. *Soil Sci.*, **63**, 79, 1947.

14. Lohnis, M. P. *Plant and Soil*, **12**, 339, 1960.

15. Mittal, O. P. and Roy, S. D. *J. Indian Soc. Soil Sci.*, **11**, 17, 1963.

16. Muhr, G. R., Datta, N. P., Money, H. S., Lelay, V. K. and Donahue, R. L. *Soil Testing in India, USAID, India, New Delhi*, 1965.

17. Piper, C. S. *Soil and Plant Analysis, Univ. of Adelaide, Australia*, 1950.

18. Randhawa, N. S., Kanwar, J. S. and Nijhawan, S. D. *Soil Sci.*, **92**, 106, 1961.

19. Sherman, G. D. and Harmer, P. M. *Soil Sci. Soc., Amer., Proc.* **7**, 398, 1943.

20. Sherman, G. D. and Fujimoto, C. K. *Ibid*, **11**, 206, 1947.

21. Sharma, S. G. and Motiramani, D. P. *J. Indian Soc. Soil Sci.*, **12**, 249, 1964.

22. Sharma, R. S. *M.Sc. (AG) Thesis of the J. N. Agri. University, Jabalpur*, 1966.

23. Sharma, O. P. and Shinde, D. A. *J. Indian Soc. Soil Sci.*, **16**, 65, 1968.

24. Toth, S. J. *Soil Sci.*, **71**, 467, 1951.

25. Vinayak, C. P., Talati, N. R. and Mathur, C. M. *J. Agri. Fd. Chem.*, **12**, 275, 1964.

26. Willard, H. H. and Greathouse, L. H. *J. Amer. Chem. Soc.*, **39**, 2366, 1917.

27. Walkley, A. and Black, I. A. *Soil Sci.*, **37**, 29, 1934.

28. Yadav, J. S. P. and Kalra, K. K. *J. Indian Soc. Soil Sci.*, **12**, 225, 1964.

29. Zende, G. K., Hiremath, K. C. and Bhadrapur, T. G. *Indian J. of Agron.*, **3**, 172, 1959.

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Studies on Antibacterials Part IV Syntheses of Naphthalene analogues of Chalcone

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Reactions of 4-hydroxy-3-nitro (or-5-bromo)-1-acetonaphthone with various aryl aldehydes in presence of 40% aqueous potassium hydroxide solution lead to the formation of coloured crystalline compounds. Some of these compounds have been found to be effective against *S. aureus*.

Chalcone compounds are known to possess antibacterial[1,2], cardiovascular[3], germicidal[7], antifungal[4], etc. activities. Davis and Eaton have suggessted[5] that the compounds having (-CH = CH - CO-) grouping possess good insecticidal properties. In the present communication we have synthesized naphthalene analogues of chalcone in which phenyl and naphthyl groups are bridged by this grouping. These compounds have been prepared by the condensation of 4-hydroxy-3-nitro (or-3-bromo) -1- acetonaphthone with various aromatic aldehydes such as, bromo, chloro, methyl, hydroxy, nitro benzaldehydes and also with bromo veratricaldehydes, bromo vanillin, nitro vanillin, piperonal, bromo piperonal and 4-dimethylamino benzaldehyde in alcoholic medium (30-40 ml.), by using cold alkaline condensation.

It was observed that 50% alkali solution gave crystalline compounds with 4-hydroxy-3-nitro -1-acetonaphthone and the required aldehydes except nitro benzaldehydes which resulted into a mixture of unknown composition. Similar results were obtained in case of 3-bromo-4-hydroxy-1-naphthyl substituted styryl ketones. In these cases the condensations were carried out with 10% potassium hydroxide solution at room temperature. This indicates that nitro benzaldehydes are sensitive towards the action of alkali in comparison with nitro methyl aryl ketones that is why with concentrated alkalis unidentifiable mixtures are obtained.

The chalcones of the 4-hydroxy-3-nitro (or 3-bromo) -1-acetonaphthone with, halo and nitro benzaldehydes were obtained in quite good yield (about 75-90%), methyl and methoxy benzaldehydes gave about (45-60%) of the theoretical yield, hydroxy benzaldehydes were obtained in very poor yield. The poor yield in case of hydroxy chalcones of naphthalene series may be on account of secondary reactions such as, resinification, etc. in presence of alkali solutions.

The presence of ketonic group in the synthesized compounds was characterized by preparing 2,4-dinitrophenylhydrazone derivatives. The analytical data and melting points are given in table 1.

The compounds of the 4-hydroxy-3-nitro-1-acetonaphthone series were screened for their antibacterial activity by observing their effect on *S. aureus*. The activity was tested by using agar-cup method and the results were compared with benzoic acid (table 2).

TABLE 1

4-hydroxy-3-nitro-1-naphthyl substituted styryl ketones

Substituents	Colour and crystal form	Crystallized from	M.P. °C	Yield %	M.F.	Found C %	Found H %	2, 4-dinitrophenyl hydrazones M. P. °C	2, 4-dinitrophenyl hydrazones N(%) found
2-bromo	Yellow globules	EtOH	132	74·7	$C_{19}H_{12}NO_4Br$	57·36	2·75	250 *d*	12·09
4-bromo	,, needles	,,	156	71·0	$C_{19}H_{12}NO_4Br$	57·09	2·83	214	12·27
3-Chloro	Lemon Yellow plates	Aq. EtOH	161	64·0	$C_{19}H_{12}NO_4Cl$	64·30	3·46	227	13·07
2, 6-dichloro	Canary yellow needles	EtOH	87	70·8	$C_{19}H_{11}NO_4Cl_2$	58·81	2·57	204	12·46
2-hydroxy	Yellow needles	EtOAc	153	32·5	$C_{19}H_{13}NO_5$	68·16	3·67	241	13·43
4-hydroxy	Yellow plates	C_6H_6	157	36·4	$C_{19}H_{13}NO_5$	68·34	3·59	210	13·61
2, 4-dihydroxy	Clusters of needles	EtOH	146	29·0	$C_{19}H_{13}NO_6$	64·74	3·85	180	12·94
2-methoxy	Canary yellow needles	,,	137	74·0	$C_{20}H_{15}NO_5$	68·86	4·16	208	13·26
2, 3-dimethoxy	Pale globules	EtOAc	171	69·8	$C_{21}H_{17}NO_6$	66 51	4·67	237	12·73
3-nitro	Yellow needles	,,	219	67·0	$C_{19}H_{12}N_2O_6$	62·42	3 38	228 *d*	15·53
4-hydroxy-3-methoxy	,,	EtOH	141	43·0	$C_{20}H_{15}NO_6$	65·69	3·93	246	12·73
5-bromo-4-hydroxy-3-methoxy	Shining needles	80% EtOH	127	52·6	$C_{20}H_{14}NO_6Br$	54·29	3·41	189	11·24
3, 4-methylenedioxy	Yellow rods	EtOH	201	78·4	$C_{20}H_{13}NO_6$	65·93	3·27	262	13·16
4 dimethylamino	Yellow needles	,,	151	63·2	$C_{21}H_{18}N_2O_4$	69·73	5·18	253	15·53
4-hydroxy-3-bromo-1-naphthyl Substituted styryl ketones									
2-bromo	Colourless needles	EtOH	134	69·5	$C_{19}H_{12}O_2Br_2$	52·62	3·06	195	9·23
4-bromo	pale needles	EtOH	128	72·0	$C_{19}H_{12}O_2Br_2$	52·48	2·73	218	8·97
2, 6-dichloro	Colourless rods	EtOAc	178	68·6	$C_{19}H_{11}O_2Cl_2Br$	54·16	3·62	235	9·52
5-bromo-2-hydroxy	,, needles	Aq. EtOH	146	51·0	$C_{19}H_{12}O_3Br_2$	51·14	2·76	251	9·13
2-methyl	Pale yellow prisms	EtOH	127	53·7	$C_{20}H_{15}O_2Br$	65·28	4·21	237	10·29
2-hydroxy-4-methoxy	Cream coloured plates	EtOH	119	45·0	$C_{20}H_{15}O_4Br$	60·19	3·98	212	9·72
5-bromo-3, 4-dimethoxy	Canary yellow globules	C_6H_6	135-6	60·0	$C_{21}H_{16}OBr_2$	51·37	3·16	188	8·56
4-dimethoxy-6-bromo-3,	Yellow needles	50% EtOH	106	62·5	$C_{21}H_{16}O_4Br_2$	51·04	3·28	229	8·23
4-hydroxy-3-methoxy-5-nitro	Yellow plates	EtOH	145	54·0	$C_{20}H_{14}NO_6Br$	54·12	3·23	230-2	11·37
6-bromo-3, 4-methylenedioxy	,, rods	EtOH	158	59·0	$C_{20}H_{12}O_4Br_2$	50·61	2·39	262 *d*	11·86

It was found that compounds having halogen atom or nitro group in the aldehyde component possessed marked antibacterial power. The halogen atom at ortho position *w. r. s.* to the conjugated system were more active than those at meta or para position.

Experimental

All melting points are uncorrected.

3-bromo-4-hydroxy-1-acetonaphthone was prepared by bromination of 4-hydroxy-1-acetonaphthone.[6]

General Method for the Synthesis of chalcones

40% aqueous solution of potassium hydroxide was gradually added with shaking to an ethanolic solution of equimolecular quantities of 4-hydroxy-3-nitro (or 3-bromo)-1-acetonaphthone and the required aldehyde. The mixture was allowed to stand for a couple of hours with shaking at intervals. It was then poured into a beaker containing a mixture of crushed ice and water and then neutralized with 20% hydrochloric acid to a pH of 6. The precipitated compound was collected on a Buchner funnel, washed with 1% sodium bicarbonate solution to eliminate the acid formed due to Cannizaro reaction and then finally from water. The crude compound on crystallization from a suitable solvent gave crystalline compounds (table 1).

TABLE 2

Antibacterial activity of 4-hydroxy-3-nitro-1-naphthyl substituted styryl ketones

Substituents	Conc. in mg./ml.	Diam. of Zone of inhib. in mm	Substituents	Conc. in mg./ml.	Diam. of Zone of inhib. in mm
2-bromo	18	12	3-nitro	20	10
4-bromo	20	11	4-hydroxy-3-methoxy	23	–
3-chloro	19	10	2-methoxy	24	8
2, 6-dichloro	18	12	3, 4-methylene-dioxy	23	6
2-hydroxy	24	–	4-dimethylamino	22	–
4-hydroxy	23	–			
2, 4-dihydroxy	25	–			
2-methoxy	27	–			
2, 3-dimethoxy	26	–			
Benzoic acid	20	12			

References

1. Schraufstatter. *Experientia*, **4**, 484-5, 1948.
2. Schraufstatter and Deutsch. *Naturforsch*, **3***b*, 163-71, 1948.
3. Devitt, Timoney and Vickars. *J. Org. Chem.*, **26**, 4941, 1961.
4. Mc. Gowan and Brian. *Chem. Abstr.*, **49**, 9334*b*, 1949.
5. Davis and Eaton. *Ann. Applied Bio.*, **37**, 471-89, 1950.
6. Akram, M. and Desai, R. D. *Proc. Indian Acad., Sci.*, **11A**, 149-55, 1940.
7. Misra, S. S., Kushwaha, S. C. and Lal, J. B. *Proc. Nat. Acad. Sci.* (*In press*)

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Intrinsic properties of steady magnetogasdynamic flows II

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Abstract

Herein we have studied the variations of flow parameters for plane hydromagnetic flows along the streamlines and their orthogonal trajectories; considering two systems of plane curves one related to the streamlines and the other to their orthogonal trajectories.

1. Introduction

Warsic [5] in his recent paper has studied the variation of flow parameters for steady plane magnetogasdynamic flows. Defining two systems of plane orthogonal curves of congruences one related to the magnetic lines of force in earlier paper [4] we have studied the intrinsic properties of plane hydromagnetic flows. In this paper we have studied the variations of flow parameters along the streamlines and their orthogonal trajectories for plane hydromagnetic flows. Decomposing the equation governing conservation of mass into intrinsic form, it is proved that the streamline patterns as a system of concentric circles or parallel lines, if the momentum per unit mass of fluid is conserved along a streamline, which is independent of the magnetic field. It is also observed that the coincidence of isovels with the streamlines does not imply the coincidence of isobars with them as in the case of non-magnetic flows. The field equation is independent of the tangentral component of the magnetic field to the streamline. Introducing the velocity of the sound, we have transformed the equation into intrinsic form and observed that the variation of mach number along a streamline is equal to the product of the curvature of normal congruence and the mach number for Chaplygon's gas.

2. (*A*) *Basic Equations*

The basic equations governing steady hydromagnetic flows, in the absence of extraneous forces are given below in the usual notation [2] :

$$\text{(1)} \qquad \text{div}\,(\rho \vec{q}\,) = 0$$

$$\rho\,(\vec{q}\cdot\nabla)\,\vec{q} = -\,\nabla p + \mu e\,\vec{J}\;\lambda\;\vec{H} \tag{2}$$

$$(\vec{q}\cdot\nabla)\,\vec{H} + \vec{H}\,\mathrm{div}\,\vec{q} - (\vec{H}\cdot\nabla)\,\vec{q} = 0 \tag{3}$$

$$p = \rho^{\gamma}\, e^{S/\lambda c_v} \tag{4}$$

$$\mathrm{div}\,\vec{H} = 0 \tag{5}$$

$$(\vec{q}\cdot\nabla)\,S = 0 \tag{6}$$

where $\vec{q}$, $\vec{H}$, ρ, p, S, μe, γ, λ and c_v are the velocity vector, the magnetic field, the density, the pressure, the specific entropy, the magnetic permeability, the adiabatic exponent, Joul's constant and the specific heat at constant volume respectively.

(B) *Geometric Results*

Considering $\vec{t}$ and $\vec{n}$ as two plane curves of congruence, one of them related to the streamlines and the other to their orthogonal trajectories and also denoting $\left(\frac{d}{ds}, \frac{d}{dn}\right)$ as the directional derivatives along these vectors, we have the following geometric results [3] :

$$\frac{d\vec{\gamma}}{ds} = \vec{t} = \frac{\vec{q}}{q} \tag{7}$$

$$\frac{d\vec{t}}{ds} = \vec{n}\,k \tag{8}$$

$$\frac{d\vec{n}}{ds} = -\,\vec{t}\,k \tag{9}$$

$$\frac{d\vec{n}}{dn} = +\,\vec{t}\,k' \tag{10}$$

$$\frac{d\vec{t}}{dn} = -\,\vec{n}\,k' \tag{11}$$

where k and k' are the curvatures of the streamlines and their orthogonal trajectories respectively.

Also using the irrotational property of '∇f' we obtain

$$k'\,\frac{df}{dn} + \frac{d}{dn}\left(\frac{df}{ds}\right) = k\,\frac{df}{ds} + \frac{d}{ds}\left(\frac{df}{dn}\right) \tag{12}$$

This has to be satisfied by any flow quantity.

3. *Decomposition into Intrinsic Form* :

In this section, we shall make use of the above geometric results to transform the basic equations into intrinsic form and study some of the kinetic and kinematic properties of fluid flows.

Using (7) in (1) we have

$$k' = \frac{d}{ds} \log (\rho\, q) \tag{13}$$

which gives the curvature of orthogonal trajectories of the streamlines. If the momentum of a unit fluid mass is conserved along an individual streamline, then the stream patterns must be either a system of concentric circles or a family of parallel straight lines, *i.e.*, the flow due to a vortex type or rectilinear.

Forming the scalar product of (2) by $\vec{t}$ and $\vec{n}$, we obtain successively

$$\rho\, q \frac{dq}{ds} = -\frac{dp}{ds} - \mu_e J H_n \tag{14}$$

$$\rho\, q^2 k = -\frac{dp}{dn} + \mu_e J H_t \tag{15}$$

where H_t and H_n are the resolved parts of the magnetic field along the streamlines and their orthogonal trajectories. From (14) we observe that the coincidence of either isovels or isobars with the streamlines does not imply the remaining as in the case of nonmagnetic flows. *c.f.* Nemenyi and Prim [1]. Also it is evident from (14) that the coincidence of any two of isovels, isobars and the magnetic lines with the streamlines implies the remaining. From (15) we obtain the current density expression, when the streamlines and isobars intersect orthogonally, *i.e.*, the pressure remains uniform along the orthogonal trajectories of the streamlines as

$$J = \frac{\rho\, k\, q^2}{\mu_e H_t} \tag{16}$$

Also if the streamlines are straight, the magnetic lines are either circles or straight lines.

The field equation (3) in intrinsic form can be decomposed as

$$\begin{gathered} q\left(\frac{dH_t}{ds} - k\, H_n\right) + H_t \left(\frac{dq}{ds} - k\, q\right) \\ -\left(H_t \frac{dq}{ds} + H_n \left(\frac{dq}{dn}\right)\right) = 0 \end{gathered} \tag{17}$$

$$\begin{gathered} q\left(kH_t + \frac{dH_n}{ds}\right) + H_n \left(\frac{dq}{ds} - k'\, q\right) \\ -(kq\, H_t - k'q\, H_n) = 0 \end{gathered} \tag{18}$$

The conservation of magnetic field can be written as

$$\frac{dH_t}{ds} + \frac{dH_n}{dn} - (k\, H_n + k'\, H_t) = 0 \tag{19}$$

Eliminating $\frac{dH_t}{ds}$ from (17) and (19) we obtain

$$\frac{d}{ds} (q\, H_n) = 0 \tag{20}$$

Also eliminating $\frac{dH_n}{ds}$ from (18) and (19) we obtain

$$\frac{d}{dn}(q\, H_n) = 0 \tag{21}$$

Eliminating 'q' and H_n from (20) and (21) we obtain

$$\frac{dH_n}{ds}\frac{dq}{dn} - \frac{dq}{ds}\frac{dH_n}{dn} = 0 \tag{22}$$

This is the field equation in intrinsic form

Using (4), (2) can be written as

$$(\vec{q}\cdot\nabla)\vec{q} + \frac{2c}{\gamma-1}\nabla c = \frac{c^2}{\lambda c_p(\gamma-1)}\nabla S + \frac{\mu_e}{\rho}\vec{J}\,\lambda\,\vec{H} \tag{23}$$

Forming scalar product of (23) by $\vec{t}, \vec{n}$ and using (4) we get

$$q\frac{dq}{ds} + \frac{2c}{\gamma-1}\frac{dc}{ds} = -\frac{\mu_e}{\rho}\,J H_n \tag{24}$$

$$k\, q^2 + \frac{2c}{\gamma-1}\frac{dc}{dn} = \frac{c^2}{\lambda c_p(\gamma-1)}\frac{dS}{dn} + \frac{\mu_e}{\rho}\,J H_t \tag{25}$$

From (14) and (24) we conclude that the coincidence of any two of the isovels, isobars, magnetic lines and sound vels with the streamlines implies the remaining two.

Also using (4) in (1) we obtain

$$k' = \frac{d}{ds}\log M + \frac{\gamma+1}{\gamma-1}\frac{d}{ds}\log c \tag{26}$$

Eliminating the density from (1) and (4) we obtain

$$k' = \frac{d}{ds}\log\left(c^{\frac{2}{\gamma-1}}\, q\right) \tag{27}$$

For Chaplygin's gas (27) simplifies to

$$k' = \frac{d}{ds}\log M \tag{28}$$

This gives the variation of the mach number along a streamline.

References

1. Nemenyi, P. and Prim, R. C. *Joun. Math. and Phys. Mars. Inst. Tech.*, **27**, 130, 1948.
2. Pai, S. I. Magnetogasdynamics and Plasmadynamics. *Springer Verlag OHG. Vienna*, 23, 1965.
3. Purushotham, G. *App. Sci. Res.*, **15**, 23, 1965.
4. Purushotham, G. and Sambasiva Rao, P. Intrinsic Properties of Steady Plane Magnetogasdynamic Flows I. (to appear in Defence Science Journal, India)
5. Warsic, N. A. Tensor, N. S., **15**(3), 275, 1964.

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Kinetics of oxidation of Toluene by Peroxydisulphate Ion

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Abstract

The oxidation of toluene has been studied in sulphuric acid medium. The reaction is first order w. r. t. peroxydisulphate and zero order w. r. t. toluene. The specific rate is related to $AgNO_3$ concentration by the relationship :

$$k = 0{\cdot}06 \times 10^{-3} + 1{\cdot}3\, C_{Ag^+}$$

The salt effect is negative. The energy parameters are $\triangle E = 12{\cdot}4$ Kcal/mole, $A = 7{\cdot}67 \times 10^3$ litre $mole^{-1} sec^{-1}$ and $\triangle S = -42{\cdot}85$ E. U.

The uncatalysed oxidation of different organic substrates by peroxydisulphate ion is generally slow. They are appreciably catalysed by silver ion. The kinetic studies involving peroxydisulphate ion as an oxidising agent have been recently reviewed by House[1]. Bacon and Doggart[2] found that toluene on oxidation by peroxydisulphate ion yields a number of products—benzoic acid, benzaldehyde, bibenzyl and a small amount of resinous products. However, toluene on oxidation with permangnate in aqueous medium yields benzoic acid as the chief product[3], but V^V oxidation of toluene gives benzaldehyde as the main product[4]. A review of the literature, however, shows that the kinetic study of the oxidation of toluene by peroxydisulphate ion has not been investigated so far. The present paper deals with the kinetic study of the Ag^+ catalysed oxidation of toluene by peroxydisulphate ion.

Experimental

The experimental procedure is similar to that adopted in the studies on phthalic acid-peroxydisulphate reaction[5]. All chemicals used were of Analar grade and toluene of Analar grade was used after redistillation.

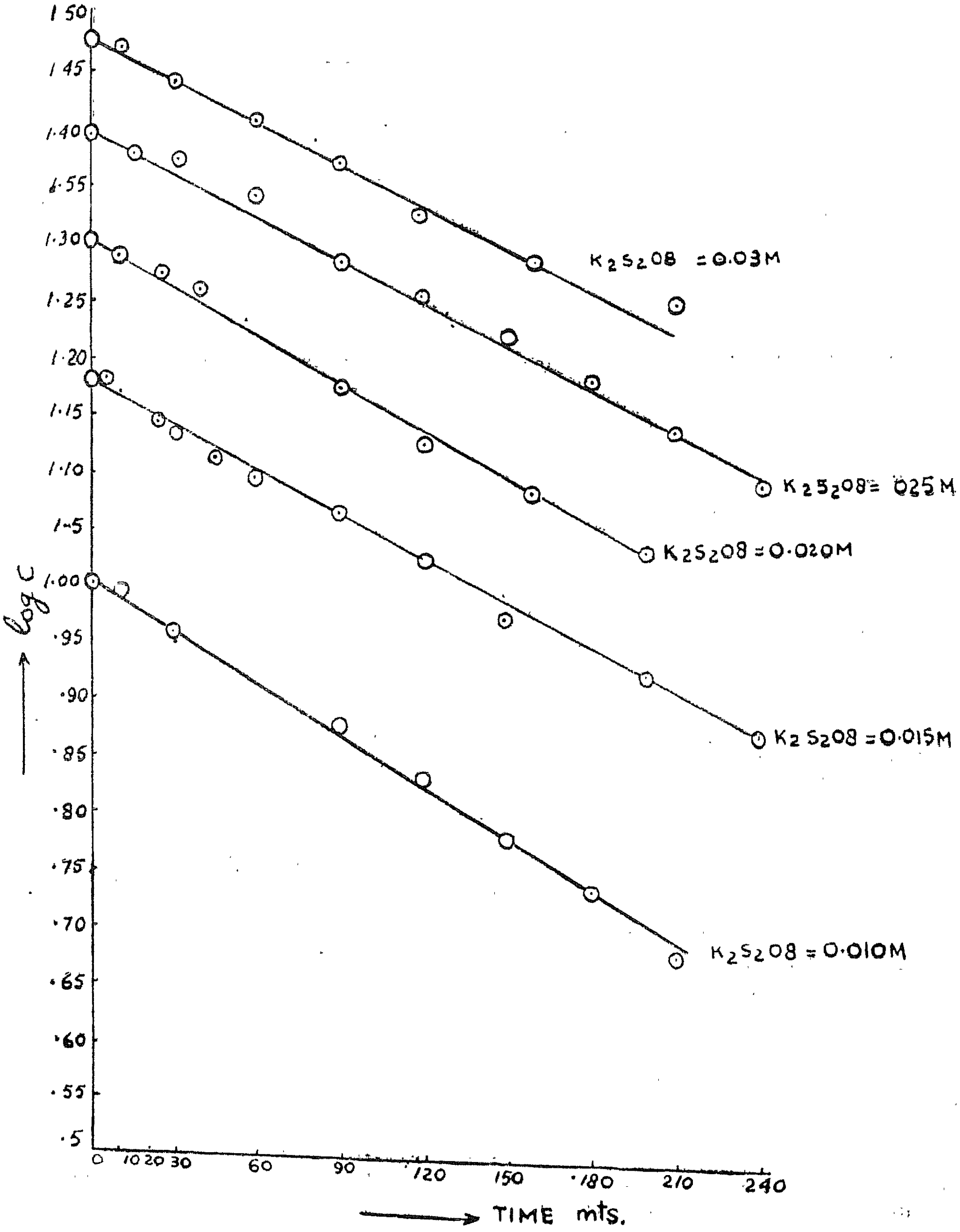
The reaction was slow but the rate was measurable after addition of $AgNO_3$ at 45°C and above.

Results and Measurements

Since the self decomposition of peroxydisulphate ion at this temperature is appreciable in the presence of $AgNO_3$, the rate of self decomposition was studied simultaneously and the rate of disappearance of S_2O_8'' due to toluene was evaluated by subtracting the rate constant of self decomposition from the observed rate. To avoid precipitate and emulsion formation, the reaction was carried out in sulphuric acid medium (0·025 M).

Fig I

$AgNO_3$ = 0·001M
H_2SO_4 = 0·025M
Toluene = 0·057M



Effect of peroxydisulphate ion concentration

The effect of initial concentration of $K_2S_2O_8$ on its rate of disappearance was determined at constant concentrations of toluene, $AgNO_3$ and H_2SO_4, *vide* Table 1.

TABLE 1

$AgNO_3$ = 0·001M　　Toluene = 0·057M
H_2SO_4 = 0·025M　　Temp. = 45°C.

S. No.	Concn. of $K_2S_2O_8$ (M)	$k' \times 10^3$ min^{-1}	$k'' \times 10^3$ min^{-1}	$(k'' - k') \times 10^3$ min^{-1}
1	0·005	1·76	3·90	2·14
2	0·010	1·92	3·44	1·52
3	0·015	2·01	3·34	1·33
4	0·020	2·19	3·11	0·92
5	0.025	2·23	2·79	0·56
6	0·030	2·24	2·73	0·49

k'' = Specific rate in presence of toluene
k' = Specific rate in absence of toluene

It is seen that at all concentrations of peroxydisulphate ion studied, the reaction is first order. However, the rate constant is found to decrease with the increase in S_2O_8'' concentration. This decrease could be ascribed due to increase in ionic strength but this is not possible because in any particular run the rate constant does not decrease with time although the ionic strength of the reaction mixture increases as the reaction proceeds. The other possibility for this behaviour is that the K^+ exerts a specific inhibitory effect. This type of behaviour has also been shown to be present in other peroxydisulphate redox reactions, *C.f.* Khulbe and Srivastava's[6] observations on the oxidation of glucose by $K_2S_2O_8$.

Effect of toluene concentration :

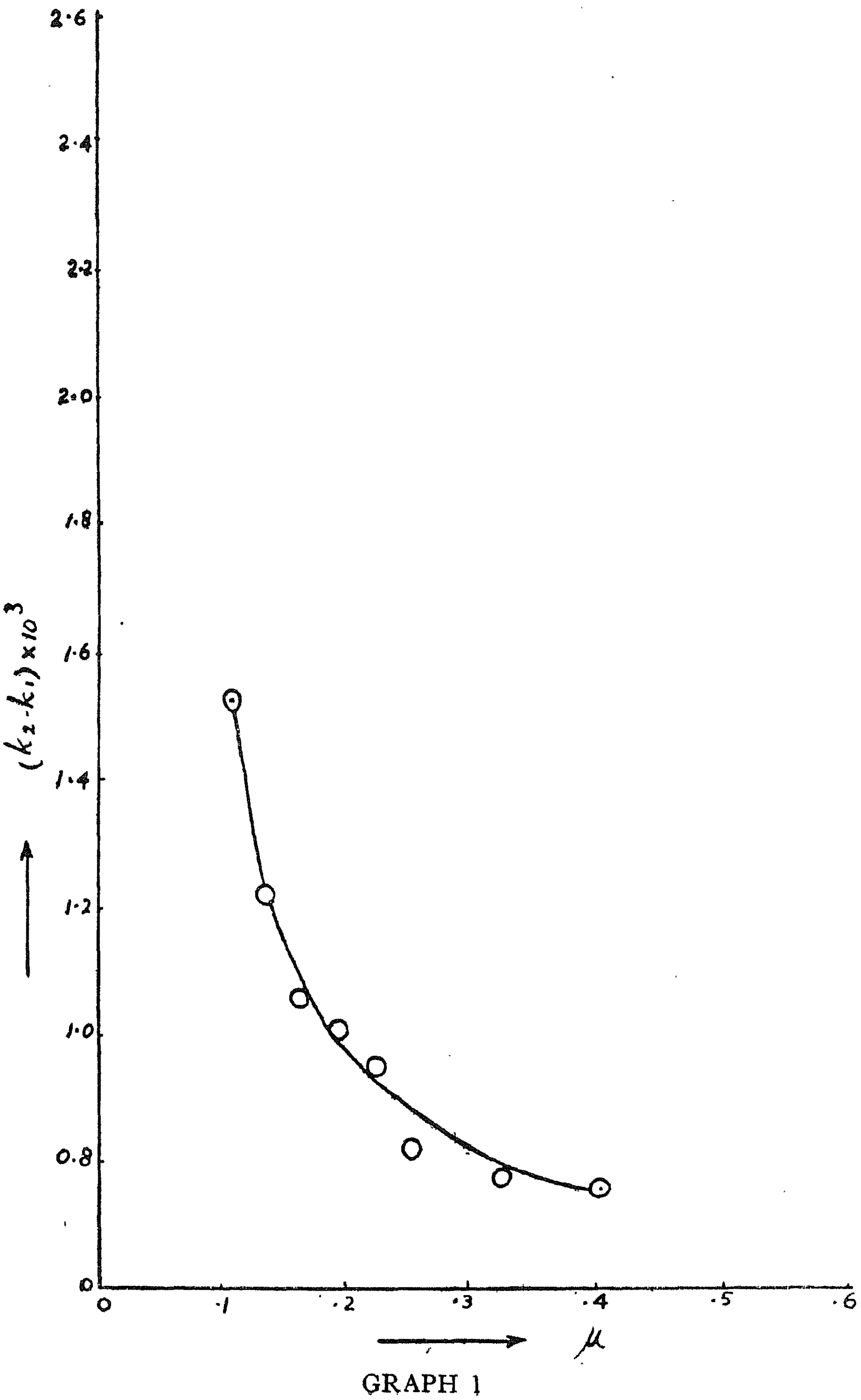
The rate constants corresponding to different initial concentration of toluene are given in Table 2.

TABLE 2

$K_2S_2O_8$ = 0·010M　　$AgNO_3$ = 0·001M
H_2SO_4 = 0·025M　　Temp. = 45°C

Concn. of toluene (M)	0·038	0·057	0·076	0·114	0·142
$(k''-k') \times 10^3$ min^{-1}	1·51	1·52	1·68	1·55	1·67

$k' = 1{\cdot}92 \times 10^{-3}\ min^{-1}$.



GRAPH 1

It is clear from the above table that the rate constant is not appreciably affected by a change in toluene concentration.

Since the reaction follows a first order behaviour when the concentration of toluene is in excess (*vide* kinetic runs 1 and 4 and 5 under Tables 1 and 2 respectively) and also when the concentration of $S_2O_8^{2-}$ and toluene are similar, the reaction, therefore, is first order in $S_2O_8^{2-}$ and zero order in toluene.

Effect of $AgNO_3$ concentration :

Table 3 summarises the results for the effect of $AgNO_3$ concentration on the rate of oxidation of toluene.

TABLE 3

$K_2S_2O_8$ = 0·010M H_2SO_4 = 0·025M
Toluene = 0·057M Temp. = 45°C

S. No.	$AgNO_3$ concn. (M)	$k' \times 10^3$ min^{-1}	$k'' \times 10^3$ min^{-1}	$(k''-k') \times 10^3$ min^{-1}
1	0·0005	1·38	2·06	0·68
2	0·0010	1·92	3·44	1·52
3	0·0015	3·19	4·98	1·79
4	0·0020	4·00	6·08	2·08
5	0·0025	4·83	8·28	3·45

The specific rate is found to be linearly related to the concentration of $AgNO_3$ and is given by the expression :

$$k = 0{\cdot}06 \times 10^{-3} + 1{\cdot}3\, C_{Ag^+}$$

Effect of temperature :

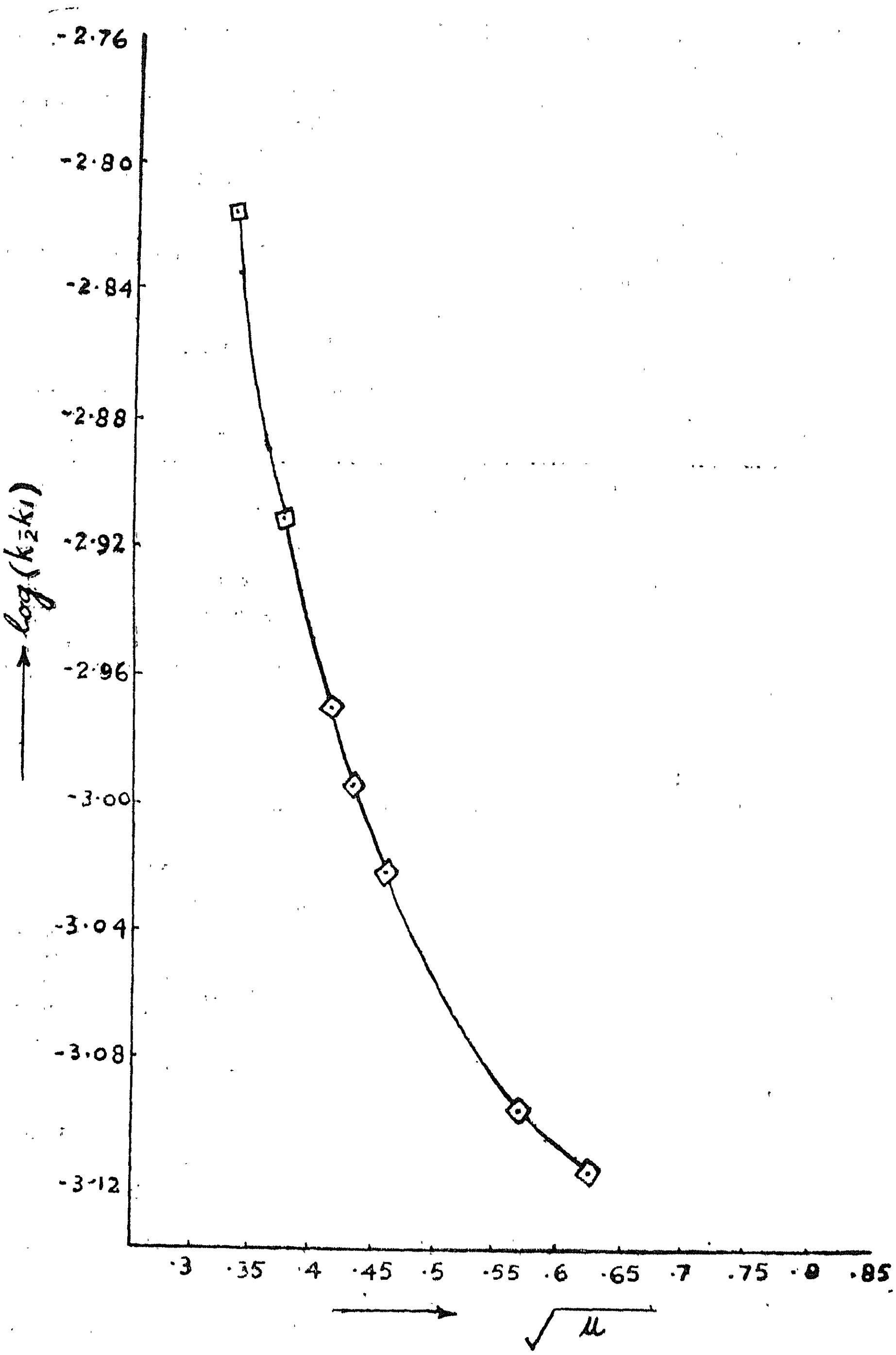
Results for kinetic runs at five different tempertures are tabulated in Table 4.

TABLE 4

$K_2S_2O_8$ = 0·010M $AgNO_3$ = 0·001M
H_2SO_4 = 0·025M Toluene = 0·057M

S. No.	Temp. °A	$k' \times 10^3$ min^{-1}	$k'' \times 10^3$ min^{-1}	$(k''-k') \times 10^3$ min^{-1}	Temp. coefficient	ΔE Kcals/mole	$A \times 10^{-3}$ litre mole^{-1} sec^{-1}	ΔS E. u.
1	308	1·12	1·91	0·79			7·48	-42·87
2	313	1·48	2·64	1·16	1·92	12·7	7·89	-42·76
3	318	1·92	3·44	1·52	1·85	12·2	7·61	-42·84
4	323	2·60	4·72	2·12	1·80	12·3	7·82	-42·86
5	328	3·69	6·44	2·75			7·56	-42·95
				Mean	1·86	12·4	$7{\cdot}67 \times 10^3$	-42·85

Δ·E for self decomposition = 12·1 Kcals mole^{-1}


-2·76
-2·80
-2·84
-2·88
-2·92
-2·96
-3·00
-3·04
-3·08
-3·12
log (k2=k1)
·3
·35
·4
·45
·5
·55
·6
·65
·7
·75
·8
·85
√μ

GRAPH 2

A perusal of the above values shows that the energy of activation for themal decomposition of S_2O_8'' and for toluene oxidation is of the same order suggesting that the mechanism of both the processes is similar.

A large negative value of $\triangle S$ suggests that the activated complex formed with the reductant is rather rigid in structure and further the rate determining reaction is between two oppositely charged ions. It may be mentioned here that a negative value for the entropy of activation has also been reported for other Ag^+ catalysed redox reactions of peroxydisulphate involving organic substrates *C.f.* Khulbe and Srivastava[7].

Salt effect :

In order to determine the effect of change of ionic strength on the rate, the reaction was carried out in the presence of different concentrations of K_2SO_4, added as neutral salt.

TABLE 5

$K_2S_2O_8$ = 0·010M, $AgNO_3$ = 0·001M, Temp. = 45°C
H_2SO_4 = 0·025M, Toluene = 0·057M

S. No.	K_2SO_4 concn. (M)	$k' \times 10^2$ min^{-1}	$k'' \times 10^3$ min^{-1}	$(k''-k') \times 10^3$ min^{-1}	log $(k''-k')$	μ	$\mu^{\frac{1}{2}}$
1	0·000	1·92	3·44	1·52	-2·817	0·106	0·326
2	0·010	1·80	3·02	1·22	-2·914	0·136	0·369
3	0·020	1·75	2·81	1·06	-2·975	0·166	0·407
4	0·030	1·70	2·71	1·01	-2·994	0·196	0·443
5	0·040	1·69	2·65	0·96	-3·019	0·226	0·475
6	0·050	1·67	2·51	0·84	-3·079	0·256	0·502
7	0·075	1·41	2·21	0·80	-3·097	0·331	0·575
8	0·100	1·13	1·89	0·76	-3·114	0·406	0·637

It is seen that the salt effect is negative. However, neither $(k''-k')$ nor log $(k''-k')$ decrease linearly with the increase of μ or $\mu^{\frac{1}{2}}$, therefore. it is not possible to decide the exact nature of the salt effect. It is worth mentioning here that since the reaction has to be carried out in sulphuric acid medium (0·025M), Brönsted relationship for salt effect is naturally not applicable.

Thus the reaction between potassium peroxydisulphate and toluene is first order w.r.t. S_2O_8'', zero order w.r.t. toluene and negative salt effect for the Ag^+ catalysed reaction suggests that the rate determining step may be between two oppositely charged ions.

Further work to study the kinetics of different stages and to isolate and identify different intermediate products is in progress after that only, a detailed reaction mechanism can be put forward.

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References

1. House, D. A. *Chem. Rev.*, **62**, 185, 1962.
2. Bacon, R. G. R. and Doggart, J. R. *J. C. S.*, 1332, 1960.
3. Cullis and Ladbury. *J. C. S.*, 555, 1955.
4. Radhakrishna Murti, P. S. and Pati, S. C. *Chemistry and Industry*, 1722–23, 1966.
5. Srivastava, S. P. and Singhal, S. K. *J. Indian Chem. Soc.*, **46**(8), 705, 1969.
6. Khulbe, C. P. and Srivastava, S. P. *Agra Univ. J. Res.* (*Sci.*), **15**(1), 27, 1966.
7. Khulbe, K. C. and Srivastava, S. P. *Agra Univ. J. Res.* (*Sci.*), **9**(2), 177, 1960.

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